Compound Distribution

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Suppose $F_1(\theta)$ be a probability distribution. Let the parameter θ of this distribution be regarded as a random variable having probability distribution $F_2(\alpha)$. The compounding of $F_1(\theta)$ & $F_2(\alpha)$ is symbolically expressed as:

$$F_1(\theta) \bigtriangleup_\theta F_2(\alpha)$$

It is known as compound F_1 distribution, F_2 known as the compounding distribution.

Ex Find Poisson(θ) $\bigtriangleup_\theta$ Gamma(a, b)

Sol. We have to find compound distribution if F_1 is Poisson distribution with parameter θ and θ is follow Gamma distribution with parameter (a, b)

$$\text{Gamma}(a, b) = \frac{a^b}{\Gamma(b)} x^{b-1} e^{-a\,x}, \quad x > 0,\ a, b > 0$$

$$\text{Poisson}(\theta) = \frac{e^{-\theta}\,\theta^m}{m!}, \quad m = 0, 1, 2....$$

Here the limit of θ is 0 to ∞.

So, compound distribution is obtained as

$$\int_0^\infty \frac{e^{-\theta}\,\theta^m}{m!}\ \frac{a^b}{\Gamma(b)}\ \theta^{b-1}\ e^{-a\,\theta}\ d\theta = \frac{a^b}{m!\,\Gamma(b)} \int_0^\infty \theta^{m+b-1}\ e^{-(a+1)\,\theta}\ d\theta$$

Using Gamma Function, Solve above integration

$$= \frac{a^b}{m!\,\Gamma(b)}\ \frac{\Gamma(m+b)}{(a+1)^{m+b}} = \frac{(m+b-1)!}{m!\,(b-1)!} \left(\frac{a}{a+1}\right)^b \left(\frac{1}{a+1}\right)^m$$

Put $\dfrac{a}{a+1} = p$ then $\dfrac{1}{a+1} = 1 - p = q$

So we can say that

$$f(\theta) = {}^{m+b-1}C_m\ p^b\ q^m; \quad m = 0, 1, 2, ...$$

This is the pmf of Negative Binomial Distribution so we can say that the compound distribution is Negative Binomial Distribution.

Ex Find Binomial(n, p) $\triangle_p$ Beta(a, b)

Sol. We have to find compound distribution if F_1 is Binomial distribution with parameter n and p. If p is follow Beta distribution with parameter (a, b)

$$\text{Binomial}(n\,p) = {}^nC_x\, p^x\, q^{n-x}, \quad n = 0, 1, 2....x$$

$$\text{Beta}(a, b) = \frac{1}{\beta(a\,,b)}\, p^{a-1}\,(1-p)^{b-1}, \quad 0 < p < 1,\ a, b > 0$$

Here the limit of p is $0 < p < 1$.

So, compound distribution is obtained as

$$\int_0^1 {}^nC_x\, p^x\, q^{n-x}\, \frac{1}{\beta(a\,,b)}\, p^{a-1}\,(1-p)^{b-1}\, dp = {}^nC_x\, \frac{1}{\beta(a\,,b)} \int_0^1 p^{x+a-1}\,(1-p)^{n-x+b-1}\, dp$$

Using Beta Function, Solve above integration

$$= {}^nC_x\, \frac{1}{\beta(a\,,b)}\, \beta(x+a,\, n-x+b) = {}^nC_x\, \frac{\Gamma(a+b)}{\Gamma(a)\,\Gamma(b)}\, \frac{\Gamma(a+x)\,\Gamma(n+b-x)}{\Gamma(n+a+b)}$$

$$= {}^nC_x\, \frac{a\,(a+1)\,(a+2), \ldots.., (a+x-1)\, b\,(b+1) \ldots..(b+n-x-1)}{(a+b)\,(a+b+1)...(a+b+n-1)}$$

So we can say that

$$f(p) = {}^nC_x\, \frac{a\,(a+1)\,(a+2), \ldots.., (a+x-1)\, b\,(b+1) \ldots..(b+n-x-1)}{(a+b)\,(a+b+1)...(a+b+n-1)}; \quad x = 0, 1, 2, ...$$

This is the pmf of Polya- Eggenberger (Beta Binomial) Distribution so we can say that the compound distribution is Polya- Eggenberger (Beta Binomial) Distribution.

Ex Find Binomial(n, p) $\triangle_p$ Beta(a, b)

Sol. We have to find compound distribution if F_1 is Binomial distribution with parameter n and p. If p is follow Beta distribution with parameter (a, b)

$$\text{Binomial}(n\,p) = {}^nC_x\, p^x\, q^{n-x}, \quad n = 0, 1, 2....x$$

$$\text{Beta}(a, b) = \frac{1}{\beta(a\,,b)}\, p^{a-1}\,(1-p)^{b-1}, \quad 0 < p < 1,\; a, b > 0$$

Here the limit of p is $0 < p < 1$.

So, compound distribution is obtained as

$$\int_0^1 {}^nC_x\; p^x\; q^{n-x}\; \frac{1}{\beta(a\,,b)}\, p^{a-1}\,(1-p)^{b-1}\; dp = {}^nC_x\; \frac{1}{\beta(a\,,b)} \int_0^1 p^{x+a-1}\,(1-p)^{n-x+b-1}\; dp$$

Using Beta Function, Solve above integration

$$= {}^nC_x\; \frac{1}{\beta(a\,,b)}\; \beta(x+a,\, n-x+b) = {}^nC_x\; \frac{\Gamma(a+b)}{\Gamma(a)\,\Gamma(b)}\; \frac{\Gamma(a+x)\,\Gamma(n+b-x)}{\Gamma(n+a+b)}$$

$$= {}^nC_x\; \frac{a\,(a+1)\,(a+2), ..., (a+x-1)\,b\,(b+1)\,.....(b+n-x-1)}{(a+b)\,(a+b+1)...(a+b+n-1)}$$

So we can say that

$$f(p) = {}^nC_x\; \frac{a\,(a+1)\,(a+2), ..., (a+x-1)\,b\,(b+1)\,.....(b+n-x-1)}{(a+b)\,(a+b+1)...(a+b+n-1)}; \quad x = 0, 1, 2, ...$$

This is the pmf of Polya- Eggenberger (Beta Binomial) Distribution so we can say that the compound distribution is Polya- Eggenberger (Beta Binomial) Distribution.

Ex Find Binomial(n, p) $\triangle_n$ Poisson(θ)

Sol. We have to find compound distribution if F_1 is Binomial distribution with parameter n and p. If n is follow Poisson distribution with parameter (θ)

$$\text{Binomial}(n\,p) = {}^nC_x\, p^x\, q^{n-x}, \quad n = 0, 1, 2....x$$

$$\text{Poisson}(\theta) = \frac{e^{-\theta}\,\theta^n}{n!}, \quad n = 0, 1, 2....$$

Here the limit of n is $n = 0, 1, 2....x$.

So, compound distribution is obtained as

$$\sum_{n=0}^{x} {}^{n}C_x \; p^x \, q^{n-x} \; \frac{e^{-\theta}\,\theta^n}{n!} \; = \sum_{n=0}^{x} \frac{n!}{x!\,(n-x)!} \; p^x \, q^{n-x} \; \frac{e^{-\theta}\,\theta^n}{n!}$$

$$= \frac{p^x \, q^{-x} \, e^{-\theta}}{x!} \sum_{n=x}^{\infty} \frac{(1-p)^n \; \theta^n}{(n-x)!} \; = \frac{p^x \, q^{-x} \, e^{-\theta} \; (\theta \, q)^x}{x!} \sum_{n=x}^{\infty} \frac{(\theta\,(1-p))^{(n-x)}}{(n-x)!}$$

$$= \frac{p^x \, \theta^x \, e^{-\theta} \; e^{\theta \, q}}{x!} \; = \frac{p^x \, \theta^x \, e^{-\theta \, p}}{x!}$$

So we can say that

$$f(n) = \frac{p^x \, \theta^x \, e^{-\theta} \; e^{\theta \, q}}{x!} \; = \frac{p^x \, \theta^x \, e^{-\theta \, p}}{x!} \qquad x = 0, 1, 2, ...$$

Assignment

1. Find Poisson(θ) $\triangle_{\theta/\phi=j}$ Poisson(α)

2. Find Poisson(θ) $\triangle_{\theta}$ Exp(α)

3. Find bin(n, p) $\triangle_{n}$ bin(m, q)

4. Find exp(θ) $\triangle_{\theta}$ U(0, 1)

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