Mathematical Expectation and Moment Generating Function

MATHEMATICAL EXPECTATION

The expected value of the random variable X or the expectation of the random variable X, denoted by $E(X)$ is defined by

$$E(X) = \begin{cases} \displaystyle\int x\, f_X(x)dx, & \text{if } x \text{ is a cont. r.v.,} \\ \displaystyle\sum x\, p(x), & \text{if } x \text{ is a disc. r.v..} \end{cases}$$

Ex. What is expected value of the number of points obtained in a single throw with an ordinary dice?

Sol. Here the random variable X is the number of points of the dice which assumes the values 1,2,3,4,5,6 each with a probability $\frac{1}{6}$

$X = x$	0	1	2	3	4	5	6
$P(X = x)$	$\frac{1}{6}$	$\frac{1}{6}$	$\frac{1}{6}$	$\frac{1}{6}$	$\frac{1}{6}$	$\frac{1}{6}$	$\frac{1}{6}$

$E(X) = \sum x\, p(x) = 1 \times \frac{1}{6} + 2 \times \frac{1}{6} + 3 \times \frac{1}{6} + 4 \times \frac{1}{6} + 5 \times \frac{1}{6} + 6 \times \frac{1}{6}$

$= (1 + 2 + 3 + 4 + 5 + 6) \times \frac{1}{6} = \frac{21}{6}$

Ex. Suppose the random variable X takes the values 0,1,2,...with probability mass function

$$p(x) = \frac{e^{-\lambda}\, \lambda^x}{x!}$$

Sol. We know that $E(X) = \displaystyle\sum_{x=0}^{\infty} x\, p(x) = \sum_{x=0}^{\infty} x\, \frac{e^{-\lambda}\, \lambda^x}{x!}$

$= \lambda\, e^{-\lambda} \displaystyle\sum_{x=1}^{\infty} \frac{\lambda^{(x-1)}}{(x-1)!} = \lambda\, e^{-\lambda}\, e^{\lambda} = \lambda$

Ex. Suppose the random variable X with probability density function $f(x) = \lambda\, e^{-\lambda\, x}, \quad x > 0$

Sol. We know that $E(X) = \displaystyle\int_0^{\infty} x\, f(x)dx = \int_0^{\infty} x\, \lambda\, e^{-\lambda\, x} dx =$

$= \lambda \displaystyle\int_0^{\infty} x\, e^{-\lambda\, x} dx = \lambda \int_0^{\infty} x^{(2-1)}\, e^{-\lambda\, x} dx = \frac{\lambda}{\lambda^2} = \frac{1}{\lambda}$

Th. If a and b are constants, then $E(a\,X+b)=a\,E(X)+b$

Prof. $E(X)=\sum(a\,x+b)\,p(x)=a\sum x\,p(x)+b\sum p(x)=a\,E(X)+b.$

Th. The expectation of the sum of two random variable X and Y is equal to the sum of their expectation.

$$E(X+Y)=E(X)+E(Y).$$

Prof. $E(X+Y)=\int\int(x+y)f(x,\,y)dx\,dy$

$=\int\int x\,f(x,\,y)dx\,dy+\int\int y\,f(x,\,y)dx\,dy$

$=\int x\int f(x,\,y)dy\,dx+\int y\int f(x,\,y)dx\,dy$

$=\int x\,g(x)\,dx+\int y\,h(y)\,dy=E(X)+E(Y)$

so, $E(X+Y)=E(X)+E(Y).$

Th. If X and Y be two independent random variable, then

$$E(X\,Y)=E(X)\,E(Y).$$

Prof. Since X and Y be two independent random variable, then $f(x,\,y)=f(x)\,f(y)$

$E(X\,Y)=\int\int x\,y\,f(x,\,y)dx\,dy=\int\int x\,y\,f(x)\,f(y)dx\,dy$

$=\int x\,f(x)dx\int y\,f(y)dy=E(X)\,E(Y).$

so $E(X\,Y)=E(X)\,E(Y).$

Expectation of a Linear Combination of Random Variables : Suppose $Y_1,Y_2,...,Y_m$ be any m random variables and if $b_1,b_2,....,b_m$ are any m constants, then

$$E\Big(\sum_{j=1}^{m}b_j\,Y_j\Big)=\sum_{j=1}^{m}b_jE(Y_j)$$

MATHEMATICAL EXPECTATION of a Function of a Random Variable

The expected value of a function of a random variable denoted by $E(h(X))$ is defined by

$$E(h(X))=\begin{cases}\int h(x)\,f_X(x)dx, & \text{if } h(x) \text{ is a cont. r.v.,}\\ \sum h(x)\,p(x), & \text{if } h(x) \text{ is a disc. r.v..}\end{cases}$$

VARIANCE

The variance of the random variable X or the variance of the random variable X, denoted by $V(X)$ is defined by

$$V(X) = \mu_2 = E(x-\bar{x})^2 = \begin{cases} \displaystyle\int (x-\bar{x})^2 \, f_X(x)dx, & \text{if } x \text{ is a cont. r.v.,} \\ \displaystyle\sum (x-\bar{x})^2 \, p(x), & \text{if } x \text{ is a disc. r.v..} \end{cases}$$

where, $\bar{x} = \mu = E(X)$

Th. $V(X) = E(X^2) - \Big(E(X)\Big)^2$

Prof. $V(X) = E(X-\bar{X})^2 = E(X^2 + \bar{X}^2 - 2\,\bar{X}\,X)$

$= E(X^2) + \bar{X}^2 - 2\,\bar{X}\,E(X) = E(X^2) + \bar{X}^2 - 2\,\bar{X}\,\bar{X} = E(X^2) - \bar{X}^2$

So, $V(X) = E(X^2) - \Big(E(X)\Big)^2$

Th. $V(a\,X+b) = a^2\,V(X)$

Prof. $Y = a\,X + b$ then $E(Y) = a\,E(X) + b$

$\Rightarrow Y - E(Y) = a(X - E(X)),$

Squaring and taking expectation of both sides, we get

$E(Y-E(Y))^2 = a^2\,E(X-E(X))^2 \Rightarrow V(Y) = a^2\,V(X)$

$\Rightarrow V(a\,X+b) = a^2\,V(X)$

Th. $V(X+Y) = V(X) + V(Y) + 2\,Cov(X,\,Y)$

Prof. $E(X+Y) = E(X) + E(Y)$ then

$\Rightarrow V(X+Y) = E\Big[(X+Y) - E(X+Y)\Big]^2$

$\Rightarrow E\Big[(X+Y) - E(X) - E(Y)\Big]^2$

$\Rightarrow E\Big[(X-E(X)) + (Y-E(Y))\Big]^2$

$\Rightarrow E\Big[(X-E(X))^2 + (Y-E(Y))^2 + 2\,(X-E(X))\,(Y-E(Y))\Big]$

$\Rightarrow E(X - E(X))^2 + E(Y - E(Y))^2 + 2\,E((X - E(X))\,(Y - E(Y)))$

$= V(X) + V(Y) + 2\,Cov(X,\,Y)$

So, $V(X + Y) = V(X) + V(Y) + 2\,Cov(X,\,Y)$

Th. $Cov(X\,,Y) = E(XY) - E(X)E(Y)$

Prof. $Cov(X\,,Y) = E\Big\{(X - E(X))(Y - E(Y))\Big\}$

$\Rightarrow E\{XY - X\,E(Y) - Y\,E(X) + E(X)E(Y)\},$

$\Rightarrow E(XY) - E(X)\,E(Y) - E(Y)\,E(X) + E(X)E(Y),$

$\Rightarrow E(XY) - E(X)\,E(Y),$

So, **Cov(X ,Y) = E(XY)- E(X) E(Y)**

If X and Y are two independent variates the $E(XY) = E(X)\,E(Y)$

$\Rightarrow Cov(X\,,Y) = E(XY) - E(X)E(Y),$

$\Rightarrow Cov(X\,,Y) = E(X)E(Y) - E(X)E(Y) \Rightarrow Cov(X\,,Y) = 0.$

So we can say that if X and Y are two independent variates then $Cov(X\,,Y) = 0$

Moment Generating Function

The Moment generating function of the random variable X denoted by $M_X(t)$ is defined by

$$M_X(t) = E(e^{t\,X}) = \begin{cases} \displaystyle\int e^{t\,x}\, f_X(x)\,dx, & \text{if } x \text{ is a cont. r.v.,} \\ \sum e^{t\,x}\,p(x), & \text{if } x \text{ is a disc. r.v..} \end{cases}$$

$M_X(t) = E(e^{t\,X}) = \sum e^{t\,x}\,p(x)$

The series expansion for $e^{t\,x}$

$$e^{t\,x} = 1 + t\,x + \frac{(t\,x)^2}{2!} + \frac{(t\,x)^3}{3!} + ... + \frac{(t\,x)^r}{r!} + ..$$

$$M_X(t) = \sum \left[1 + t\,x + \frac{(t\,x)^2}{2!} + \frac{(t\,x)^3}{3!} + ... + \frac{(t\,x)^r}{r!} + ..\right] p(x)$$

$$\Rightarrow \sum p(x) + t\sum x\,p(x) + \frac{t^2}{2!}\sum x^2\,p(x) + \frac{t^3}{3!}\sum x^3\,p(x) + ... + \frac{t^r}{r!}\sum x^r\,p(x)$$

$$\Rightarrow 1 + \mu_1'\,t + \mu_2'\,\frac{t^2}{2!} + \mu_3'\,\frac{t^3}{3!} + ... + \mu_r'\,\frac{t^r}{r!} + ...$$

where,

$\mu_r' = E(X^r) = \sum x^r\,p(x)$ is the r^{th} moment of X about origin.

Hence, we see that the co-efficient of $\frac{t^r}{r!}$ in $M_X(t)$ gives μ_r'

A simplified procedure of obtaing the few moment is using the following formula

$$\mu_r' = \left.\frac{d^r M_X(t)}{dt^r}\right|_{t=0}$$

Th. $M_{a\,X+b}(t) = e^{bt}\,M_X(a\,t)$

Prof. $M_{a\,X+b}(t) = E[e^{t(a\,X+b)}] = E[e^{t(a\,X)}\,e^{b\,t}] = e^{b\,t}\,E[e^{t(a\,X)}]$

$$\Rightarrow e^{b\,t}\,E[e^{a\,t(X)}] = e^{b\,t}\,M_X(a\,t)$$

Th. If $X_1, X_2, ...X_n$ are independent random variable, then the moment generating function of their sum $(X_1 + X_2 + ... + X_n)$ is given by $M_{(X_1+X_2+...+X_n)}(t) = M_{X_1}(t)\,M_{X_2}(t).....M_{X_n}(t)$

Prof. $M_{(X_1+X_2+...+X_n)}(t) = E(e^{t\,(X_1+X_2+...+X_n)}) = E(e^{t\,X_1}\,e^{t\,X_2}...e^{t\,X_n})$

$\Rightarrow E(e^{t\,X_1})\,E(e^{t\,X_2})...E(e^{t\,X_n})$ ($\because X_1, X_2, ...X_n$ are independent)

$$M_{(X_1+X_2+...+X_n)}(t) = M_{X_1}(t)\,M_{X_2}(t).....M_{X_n}(t)$$

Ex. Let the random variable X assume the value x with the probability law: $P(X = x) = p\,q^{(x-1)}; \quad x = 1, 2, 3, 4, ...$ and $p + q = 1$. Find the m.g.f. and hence the Mean and Variance.

Sol. $M_X(t) = E(e^{t\,X}) = \sum e^{t\,x}\,P(X = x) = \sum e^{t\,x}\,p\,q^{(x-1)}$

$$\Rightarrow \frac{p}{q}\sum (q\,e^t)^x = \frac{p}{q}\left[q\,e^t + (q\,e^t)^2 + ...\right]$$

$$\Rightarrow \frac{p}{q}\,q\,e^t\left[1 + q\,e^t + (q\,e^t)^2 + ...\right] = \frac{p\,e^t}{1 - q\,e^t}$$

So, $M_X(t) = \dfrac{p\,e^t}{1 - q\,e^t}$ and $\mu_r' = \left.\dfrac{d^r M_X(t)}{dt^r}\right|_{t=0}$

We obtain $\mu_1' = \left.\dfrac{dM_X(t)}{dt}\right|_{t=0}$

$$\mu_1' = \left.\frac{p\,e^t}{(1-q\,e^t)^2}\right|_{t=0} = \frac{p}{(1-q)^2} = \frac{1}{p}, \quad \text{so Mean} = \frac{1}{p}$$

$$\mu_2' = \left.\frac{p\,e^t\,(1+q\,e^t)}{(1-q\,e^t)^3}\right|_{t=0} = \frac{p\,(1+q)}{(1-q)^3} = \frac{(1+q)}{p^2}$$

$$\text{so Variance } \mu_2 = \mu_2' - (\mu_1')^2 = \frac{(1+q)}{p^2} - \frac{1}{p^2} = \frac{q}{p^2}$$

Ex. Find the m.g.f of the random variable X having the probability density function

$$f(x) = \begin{cases} x & 0 \le x \le 1 \\ 2-x & 1 \le x \le 2 \\ 0 & \text{otherwise.} \end{cases}$$

Sol. $M_X(t) = E(e^{tX}) = \int e^{tx}\,f(x)\,dx$

$$\Rightarrow \int_0^1 x\,e^{tx}\,dx + \int_1^2 (2-x)\,e^{tx}\,dx = \frac{e^{2t}}{t^2} - \frac{2\,e^t}{t^2} + \frac{1}{t^2}$$

$$\Rightarrow \frac{1}{t^2}\left[1 + e^{2t} - 2\,e^t\right]$$

Now expanding $M_X(t)$, we have

$$\Rightarrow \frac{1}{t^2}\left[1 + \left(1 + 2\,t + \frac{(2\,t)^2}{2!} + \frac{(2\,t)^3}{3!} + ...\right) - 2\left(1 + t + \frac{t^3}{3!} + ...\right)\right]$$

$$\Rightarrow \frac{1}{t^2}\left[t^2 + t^3 + \frac{7}{12}\,t^4 + ...\right] = 1 + t + \frac{7}{12}\,t^2 + ...$$

$$\mu_1' = \text{Coefficient of } t \text{ in } M_X(t) = 1,$$

$$\mu_2' = \text{Coefficient of } \frac{t^2}{2!} \text{ in } M_X(t) = 2!\,\frac{7}{12} = \frac{7}{6},$$

$$\text{so Variance } \mu_2 = \mu_2' - (\mu_1')^2 = \frac{7}{6} - 1 = \frac{1}{6}$$

Assignment

1. Find the m.g.f. of the discrete random variable X that has the probability distribution $P(X = x) = 2\,(\frac{1}{3})^x; \quad x = 1, 2, 3, ...$and use it to determine the Mean and Variance.

2. Show that if a random variable has the probability density

$$f(x,\,y) = \begin{cases} \frac{1}{2}\,e^{x} & -\infty < x < 0 \\ \frac{1}{2}\,e^{-x} & 0 < x < \infty. \end{cases}$$

its M.G.F. is given by $M_X(t) = \frac{1}{1-t^2}$ and hence the Mean and Variance.

3. Compute the m.g.f. of the distribution defined by $f(x) = x\,e^{-x};\ x > 0$ and use it to determine the Mean and Variance.

4. Find M.G.F., mean and variance for the p.d.f. $f(x) = c\,e^{-x};\ x > 0$ where c is an unknown constant. Find c also.

5. Find the m.g.f. of the discrete random variable X that has the probability distribution $P(X = x) = \dfrac{\lambda^x\, e^{-\lambda}}{x!};\quad x = 0, 1, 2, 3, ...$and use it to determine the Mean and Variance.

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