Bivariate Normal Distribution

Bivariate Normal Distribution: The joint probability density function of the bivariate normal distribution of two random variable X and Y is given by:

$$f_{X,Y}(x,y) = \frac{1}{2\pi\,\sigma_x\,\sigma_y\,\sqrt{1-\rho^2}} \;\; exp\bigg\{\frac{-1}{2(1-\rho^2)}\bigg\{\Big(\frac{x-\mu_x}{\sigma_x}\Big)^2 + \Big(\frac{y-\mu_y}{\sigma_y}\Big)^2 - 2\rho\Big(\frac{x-\mu_x}{\sigma_x}\Big)\Big(\frac{y-\mu_y}{\sigma_y}\Big)\bigg\}\bigg\},$$

where

- $-\infty < x < \infty$ and $-\infty < y < \infty$
- $-\infty < \mu_x < \infty$ and $-\infty < \mu_y < \infty$ are the marginal means
- $\sigma_x > \infty$ and $\sigma_y > \infty$ are the marginal standard deviations
- $-1 < \rho < 1$ is the correlation coefficient
- $\mu_x, \mu_y, \sigma_x, \sigma_y, \rho$ are the five parameters

Marginal Distribution

The Marginal density of the variables X is given by.

$$f_X(x) = \int_{-\infty}^{\infty} f_{X,Y}(x,y) \;\; dy$$

$$f_X(x) = \int_{-\infty}^{\infty} \frac{1}{2\pi\,\sigma_x\,\sigma_y\,\sqrt{1-\rho^2}} \;\; exp\bigg\{\frac{-1}{2(1-\rho^2)}\bigg\{\Big(\frac{x-\mu_x}{\sigma_x}\Big)^2 + \Big(\frac{y-\mu_y}{\sigma_y}\Big)^2 - 2\rho\Big(\frac{x-\mu_x}{\sigma_x}\Big)\Big(\frac{y-\mu_y}{\sigma_y}\Big)\bigg\}\bigg\}dy$$

Now Suppose, $w = \dfrac{y-\mu_y}{\sigma_y}$, then $dy = \sigma_y\,dw$

$$f_X(x) = \int_{-\infty}^{\infty} \frac{1}{2\pi\,\sigma_x\,\sigma_y\,\sqrt{1-\rho^2}} \;\; exp\bigg\{\frac{-1}{2(1-\rho^2)}\bigg\{\Big(\frac{x-\mu_x}{\sigma_x}\Big)^2 + w^2 - 2\rho\,w\Big(\frac{x-\mu_x}{\sigma_x}\Big)\bigg\}\bigg\}\sigma_y\,dw$$

$$= \frac{1}{2\pi\,\sigma_x\,\sqrt{1-\rho^2}} \;\; exp\bigg\{\frac{-1}{2}\Big(\frac{x-\mu_x}{\sigma_x}\Big)^2\bigg\} \;\; \int_{-\infty}^{\infty} exp\bigg\{\frac{-1}{2(1-\rho^2)}\bigg\{w - \rho\Big(\frac{x-\mu_x}{\sigma_x}\Big)\bigg\}^2\bigg\}\,dw$$

Again substitutions $$u = \frac{w - \rho(x-\mu_x)/\sigma_x}{\sqrt{1-\rho^2}} \text{ and } du = \frac{dw}{\sqrt{1-\rho^2}}$$

$$f_X(x) = \frac{1}{2\pi\,\sigma_x\sqrt{1-\rho^2}} \quad exp\left\{\frac{-1}{2}\left(\frac{x-\mu_x}{\sigma_x}\right)^2\right\} \quad \int_{-\infty}^{\infty} e^{-\frac{u^2}{2}}\sqrt{1-\rho^2}\,du$$

$$= \frac{1}{2\pi\,\sigma_x} \quad exp\left\{\frac{-1}{2}\left(\frac{x-\mu_x}{\sigma_x}\right)^2\right\} \quad \int_{-\infty}^{\infty} e^{-\frac{u^2}{2}}\,du \qquad \text{Since } \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{-\frac{u^2}{2}}\,du = 1$$

$$= \frac{1}{2\pi\,\sigma_x} \quad exp\left\{\frac{-1}{2}\left(\frac{x-\mu_x}{\sigma_x}\right)^2\right\} \quad \sqrt{2\pi}$$

$$f_X(x) = \frac{1}{\sigma_x\sqrt{2\pi}} \quad exp\left\{\frac{-1}{2}\left(\frac{x-\mu_x}{\sigma_x}\right)^2\right\}$$

So, X follow Normal distribution with mean μ_x and S.D. σ_x, We can say that $X \sim N(\mu_x,\, \sigma_x)$.

Similarly, Y follow Normal distribution with mean μ_y and S.D. σ_y, We can say that $Y \sim N(\mu_y,\, \sigma_y)$.

Conditional Distribution

The conditional density of the variables X for fixed Y is obtained as

$$f_{X|Y}(x|y) = \frac{f_{X,Y}(x,y)}{f_Y(y)}$$

$$= \frac{\dfrac{1}{2\pi\,\sigma_x\,\sigma_y\sqrt{1-\rho^2}} \quad exp\left\{\dfrac{-1}{2(1-\rho^2)}\left\{\left(\dfrac{x-\mu_x}{\sigma_x}\right)^2 + \left(\dfrac{y-\mu_y}{\sigma_y}\right)^2 - 2\rho\left(\dfrac{x-\mu_x}{\sigma_x}\right)\left(\dfrac{y-\mu_y}{\sigma_y}\right)\right\}\right\}}{\sigma_y\sqrt{2\pi} \quad exp\left\{\dfrac{1}{2}\left(\dfrac{y-\mu_y}{\sigma_y}\right)^2\right\}}$$

$$= \frac{1}{\sigma_x\sqrt{2\pi}\sqrt{1-\rho^2}} \quad exp\left\{\frac{-1}{2(1-\rho^2)}\left\{\left(\frac{x-\mu_x}{\sigma_x}\right)^2 + (1-(1-\rho^2))\left(\frac{y-\mu_y}{\sigma_y}\right)^2 - 2\rho\left(\frac{x-\mu_x}{\sigma_x}\right)\left(\frac{y-\mu_y}{\sigma_y}\right)\right\}\right\}$$

$$= \frac{1}{\sigma_x\sqrt{2\pi}\sqrt{1-\rho^2}} \quad exp\left\{\frac{-1}{2\,\sigma_x^2\,(1-\rho^2)}\left((x-\mu_x) - \rho\,\frac{\sigma_x}{\sigma_y}\,(y-\mu_y)\right)^2\right\}$$

$$f_{X|Y}(x|y) = \frac{1}{\sigma_x \sqrt{2\pi}\,\sqrt{1-\rho^2}} \quad exp\left\{\frac{-1}{2\,\sigma_x^2\,(1-\rho^2)}\left(x - \left[\mu_x + \rho\,\frac{\sigma_x}{\sigma_y}\,(y-\mu_y)\right]\right)^2\right\}$$

So, conditional distribution $X|Y$ follow Normal distribution with mean $\left(\mu_x + \rho\,\frac{\sigma_x}{\sigma_y}\,(y-\mu_y)\right)$ and S.D. $\sigma_x\,\sqrt{1-\rho^2}$. We can say that $X|Y \sim N\left(\mu_x + \rho\,\frac{\sigma_x}{\sigma_y}\,(y-\mu_y),\ \sigma_x\,\sqrt{1-\rho^2}\right)$.

Similarly, the conditional density of the variables Y for fixed X is given by

$$f_{Y|X}(y|x) = \frac{1}{\sigma_y \sqrt{2\pi}\,\sqrt{1-\rho^2}} \quad exp\left\{\frac{-1}{2\,\sigma_y^2\,(1-\rho^2)}\left(y - \left[\mu_y + \rho\,\frac{\sigma_y}{\sigma_x}\,(x-\mu_x)\right]\right)^2\right\}$$

So, conditional distribution $X|Y$ follow Normal distribution with mean $\left(\mu_y + \rho\,\frac{\sigma_y}{\sigma_x}\,(x-\mu_x)\right)$ and S.D. $\sigma_y\,\sqrt{1-\rho^2}$. We can say that $Y|X \sim N\left(\mu_y + \rho\,\frac{\sigma_y}{\sigma_x}\,(x-\mu_x),\ \sigma_y\,\sqrt{1-\rho^2}\right)$.

Moment Generating function

The moment generating function of Bivariate Normal Distribution is obtained as

$$M_{X,Y}(t_1,\ t_2) = E(e^{(t_1\,X + t_2\,Y)}) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{(t_1\,x + t_2\,y)}\ f_{X,Y}(x,y)\quad dx\,dy$$

Now Suppose, $w = \frac{x-\mu_x}{\sigma_x}$ and $z = \frac{y-\mu_y}{\sigma_y}$, then $x = \mu_x + w\,\sigma_x$ and $y = \mu_y + z\,\sigma_y$, $\quad |J| = \sigma_x\,\sigma_y$

$$M_{X,Y}(t_1,\ t_2) \ = \ \frac{exp(t_1\,\mu_x + t_2\,\mu_y)}{2\pi\sqrt{1-\rho^2}} \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \ exp\left\{w\,t_1\,\sigma_x \ + \ z\,t_2\,\sigma_y \ - \ \frac{1}{2\,(1-\rho^2)}(w^2 \ + \ z^2 \ - 2\,w\,z\,\rho)\right\} dw\,dz$$

After solving above equation, we get

$$M_{X,Y}(t_1,\ t_2) = exp\left\{\mu_x\,t_1 + \mu_y\,t_2 + 0.5(t_1^2\,\sigma_x^2 + t_2^2\,\sigma_y^2 + 2\,\rho\,\sigma_x\,\sigma_y\,t_1\,t_2)\right\}$$

Th: If (X,Y) has a bivariate normal distribution, then X and Y are independent if and only if $\rho = 0$.

The joint probability density function of two random variable X and Y are bivariate normal distribution.

If $\rho = 0$ then

$$f_{X,Y}(x, y) = \frac{1}{2\pi\,\sigma_x\,\sigma_y} \quad exp\left\{\frac{-1}{2}\left\{\left(\frac{x-\mu_x}{\sigma_x}\right)^2 + \left(\frac{y-\mu_y}{\sigma_y}\right)^2\right\}\right\},$$

$$f_{X,Y}(x, y) = \frac{1}{\sigma_x\sqrt{2\pi}} \quad exp\left\{\frac{-1}{2}\left(\frac{x-\mu_x}{\sigma_x}\right)^2\right\} \quad \frac{1}{\sigma_y\sqrt{2\pi}} \quad exp\left\{\frac{-1}{2}\left(\frac{y-\mu_y}{\sigma_y}\right)^2\right\}$$

$$f_{X,Y}(x, y) = f_X(x)\,f_Y(y)$$

so, X and Y are independent.

Q.1: Find the mean, variance and correlation coefficient, if (X,Y) has a bivariate normal distribution with Joint probability distribution function is given in this form

$$f_{X,Y}(x, y) = e^{-8\,x^2 - 6\,x\,y - 18\,y^2}$$

Q.2: If (X,Y) has a bivariate normal distribution with $\mu_x = 100, \sigma_x = 4, \mu_y = 80, \sigma_y = 1$ and $\rho = 0.5.$ then find the

- Marginal probability density function of X.
- Marginal probability density function of Y.
- Conditional probability distribution of X given Y=100.
- Conditional probability distribution of Y given X=80.

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