Bivariate Exponential Distribution

Bivariate Exponential Distribution: The joint Distribution function of the bivariate Exponential distribution of two random variable X and Y is given by:

$$F_{X,Y}(x,y) = e^{-(\alpha_1\,x+\alpha_2\,y+\theta\,x\,y)},$$

where

- $x > 0$ and $y > 0$
- $\alpha_1 > 0, \alpha_2 > 0$ and $\theta > 0$ are the three parameters

Joint Probability Density Function:

The joint density function of the bivariate Exponential distribution of two random variables X and Y is given by.

$$f_{X,Y}(x,y) = \frac{\partial^2 F_{X,Y}(x,y)}{\partial x\,\partial y}$$

$$= \frac{\partial^2}{\partial x\,\partial y}\left\{e^{-(\alpha_1\,x+\alpha_2\,y+\theta\,x\,y)}\right\} = \frac{\partial}{\partial y}\left\{\frac{\partial}{\partial x}e^{-(\alpha_1\,x+\alpha_2\,y+\theta\,x\,y)}\right\}$$

$$= \frac{\partial}{\partial y}\left\{(-1)\,(\alpha_1+\theta\,y)\,e^{-(\alpha_1\,x+\alpha_2\,y+\theta\,x\,y)}\right\}$$

$$= (-1)\,\theta\,e^{-(\alpha_1\,x+\alpha_2\,y+\theta\,x\,y)} + (-1)\,(\alpha_1+\theta\,y)\,(-1)\,(\alpha_2+\theta\,x)\,e^{-(\alpha_1\,x+\alpha_2\,y+\theta\,x\,y)}$$

$$= e^{-(\alpha_1\,x+\alpha_2\,y+\theta\,x\,y)}\Big[(\alpha_2+\theta\,x)\,(\alpha_1+\theta\,y)\,-\theta\Big]$$

Thus,

$$f_{X,Y}(x,y) = e^{-(\alpha_1\,x+\alpha_2\,y+\theta\,x\,y)}\Big[(\alpha_2+\theta\,x)\,(\alpha_1+\theta\,y)\,-\theta\Big], \qquad x,y>0.$$

Note: If $\theta = 0$ then

$$f_{X,Y}(x,y) = \alpha_1\,\alpha_2\,e^{-(\alpha_1\,x+\alpha_2\,y)}, \qquad x,y>0,$$

$$f_{X,Y}(x,y) = \alpha_1\,e^{-\alpha_1\,x}\;\;\alpha_2\,e^{-\alpha_2\,y}, \qquad x,y>0.$$

Thus, θ is link between X and Y.

Marginal Distribution

The Marginal density of the variables X is given by.

$$f_X(x) = \int_0^\infty\; f_{X,Y}(x,y)\;\; dy$$

$$f_X(x) = \int_0^\infty\; e^{-(\alpha_1\,x+\alpha_2\,y+\theta\,x\,y)}\Big[(\alpha_2+\theta\,x)\,(\alpha_1+\theta\,y)\;-\theta\Big]dy$$

$$= \int_0^\infty (\alpha_2+\theta\,x)\,(\alpha_1+\theta\,y)\;e^{-(\alpha_1\,x+\alpha_2\,y+\theta\,x\,y)}\;dy - \int_0^\infty \theta\;e^{-(\alpha_1\,x+\alpha_2\,y+\theta\,x\,y)}\;dy$$

$$= (\alpha_2+\theta\,x)\;\;e^{-\alpha_1\,x}\;\int_0^\infty (\alpha_1+\theta\,y)\;e^{-(\alpha_2+\theta\,x)\,y}\;dy - \theta\;\;e^{-\alpha_1\,x}\;\int_0^\infty\; e^{-(\alpha_2+\theta\,x)\,y}\;dy$$

Solve above integration using Gamma function, we get

$$= \alpha_1\;\;e^{-\alpha_1\,x}\; + \frac{\theta\;e^{-\alpha_1\,x}}{(\alpha_2+\theta\,x)} - \frac{\theta\;e^{-\alpha_1\,x}}{(\alpha_2+\theta\,x)}$$

$$f_X(x) = \alpha_1\;\;e^{-\alpha_1\,x}\quad x>0.$$

So, X follow univariate exponential distribution with parameter α_1.

We can say that $X \sim Exp(\alpha_1)$.

Similarly, Y follow Univariate exponential distribution with parameter α_2.

We can say that $Y \sim Exp(\alpha_2)$.

Conditional Distribution

The conditional density of the variables X for fixed Y is obtained as

$$f_{X|Y}(x|y) = \frac{f_{X,Y}(x,y)}{f_Y(y)}$$

$$= \frac{1}{\alpha_2 \ e^{-\alpha_2 y}} \ \ e^{-(\alpha_1 x + \alpha_2 y + \theta x y)} \left[(\alpha_2 + \theta x)(\alpha_1 + \theta y) \ - \theta\right]$$

$$= (\alpha_1 + \theta y) \ \ e^{-(\alpha_1 + \theta y) x} \left[\frac{(\alpha_2 + \theta x)}{\alpha_2} - \frac{\theta}{\alpha_2 \ (\alpha_1 + \theta y)}\right]$$

So,

$$f_{X|Y}(x|y) = (\alpha_1 + \theta y) \ \ e^{-(\alpha_1 + \theta y) x} \left[\frac{(\alpha_2 + \theta x)}{\alpha_2} - \frac{\theta}{\alpha_2 \ (\alpha_1 + \theta y)}\right]$$

Similarly, the conditional density of the variables Y for fixed X is given by

$$f_{Y|X}(y|x) = (\alpha_2 + \theta x) \ \ e^{-(\alpha_2 + \theta x) y} \left[\frac{(\alpha_1 + \theta y)}{\alpha_1} - \frac{\theta}{\alpha_1 \ (\alpha_2 + \theta x)}\right]$$

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