Truncated Distribution

Truncated Distribution: To truncate means to cut when the domain of a distribution is truncated. We get truncated distribution.

Left Truncated Distribution.

Let $p(x)$ be the pmf of a discrete random variable X having its possible values $x = 0, 1, 2, ...m, m+1, m+2, ...$

We have,

$$\sum_{x=0}^{\infty} p(x) = 1,$$

We can write $\sum_{x=0}^{\infty} p(x)$ into two part, we get

$$\sum_{x=0}^{\infty} p(x) = \sum_{x=0}^{m} p(x) + \sum_{x=m+1}^{\infty} p(x),$$

So,

$$\sum_{x=0}^{m} p(x) + \sum_{x=m+1}^{\infty} p(x) = 1,$$

$$\sum_{x=m+1}^{\infty} p(x) = 1 - \sum_{x=0}^{n} p(x),$$

$$\frac{\sum_{x=m+1}^{\infty} p(x)}{\sum_{x=0}^{m} p(x)} = 1$$

$$\sum_{x=m+1}^{\infty} \frac{p(x)}{1-\sum_{x=0}^{m} p(x)} = 1$$

This means that,

$$p_1(x) = \frac{p(x)}{1-\sum_{x=0}^{m} p(x)}$$

is a probability distribution of random variable X for $x = m+1, m+2, ...$

$$p_1(x) = \frac{p(x)}{\sum_{x=m+1}^{\infty} p(x)}, \qquad x = m+1, m+2, ...$$

This is Left truncated p(x) distribution.

Right Truncated Distribution

Let $p(x)$ be the pmf of a discrete random variable X having its possible values $x = 0, 1, 2, ...n, n+1, n+2, ...$

We have,

$$\sum_{x=0}^{\infty} p(x) = 1,$$

We can write $\sum_{x=0}^{\infty} p(x)$ into two part, we get

$$\sum_{x=0}^{\infty} p(x) = \sum_{x=0}^{n} p(x) + \sum_{x=n+1}^{\infty} p(x),$$

So,

$$\sum_{x=0}^{n} p(x) + \sum_{x=n+1}^{\infty} p(x) = 1,$$

$$\sum_{x=0}^{n} p(x) = 1 - \sum_{x=n+1}^{\infty} p(x),$$

$$\frac{\sum_{x=0}^{n} p(x)}{1 - \sum_{x=n+1}^{\infty} p(x)} = 1$$

$$\sum_{x=0}^{n} \frac{p(x)}{1 - \sum_{x=n+1}^{\infty} p(x)} = 1$$

This means that,

$$p_2(x) = \frac{p(x)}{1 - \sum_{x=n+1}^{\infty} p(x)}$$

is a probability distribution of random variable X for $x = 0, 1, 2.....n$

$$p_2(x) = \frac{p(x)}{\sum_{x=0}^{n} p(x)}, \qquad x = 0, 1, 2, ..., n$$

This is Right truncated p(x) distribution.

Similar we can do with continuous distribution.

Zero- Truncated Binomial Distribution

Suppose $X \sim B(n\,, p)$ and we want to truncated from zero

$$p_1(x) = \frac{{}^nC_x \;\; p^x \;\; q^{n-x}}{\displaystyle\sum_{x=1}^{n} \; {}^nC_x \;\; p^x \;\; q^{n-x}}\,, \qquad x = 1, 2, ..., n$$

$$= \frac{{}^nC_x \;\; p^x \;\; q^{n-x}}{1 - p(0)} \qquad = \frac{{}^nC_x \;\; p^x \;\; q^{n-x}}{1 - q^n}\,, \qquad\qquad x = 1, 2, ..., n$$

This is the truncated Binomial Distribution.

Moments The r^{th} moment about origin of zero truncated Binomial distribution is obtained as,

$$\mu'_r = E(X^r) = \sum_{x=1}^{n} \; x^r \;\, p(x) = \sum_{x=0}^{n} \; x^r \;\; \frac{{}^nC_x \;\; p^x \;\; q^{n-x}}{1 - q^n}\,,$$

$$= \frac{1}{1 - q^n} \quad \sum_{x=0}^{n} \; x^r \; {}^nC_x \;\; p^x \;\; q^{n-x}\,,$$

$$= \frac{1}{1 - q^n} \;\; \times \;\; (r^{th} \text{ moment about origin of Binomial Distribution.})$$

$$\text{Mean} = \mu'_1 = E(X) = \frac{np}{1 - q^n}$$

$$\mu'_2 = E(X^2) = \frac{n(n-1)p^2 + np}{1 - q^n}$$

$$\text{Variance } \mu_2 = \mu'_2 - (\mu'_1)^2 \;\; = \;\; \frac{n(n-1)p^2 + np}{1 - q^n} - \left(\frac{np}{1 - q^n}\right)^2$$

$$\mu_2 = \frac{np}{1-q^n} \quad \left[q + np - \frac{np}{1-q^n}\right]$$

Similarly, we can find μ_3' and μ_4'. Also β_1 and β_2.

Zero- Truncated Poisson Distribution

Suppose $X \sim P(\lambda)$ and we want to truncated from zero

$$p_1(x) = \frac{p(x)}{1-p(0)} = \frac{\lambda^x \ e^{-\lambda}}{x! \left(1 - \dfrac{\lambda^0 \, e^{-\lambda}}{0!}\right)}, \qquad x = 1, 2, ..., n$$

$$= \frac{\lambda^x \ e^{-\lambda}}{x! \ (1-e^{-\lambda})}, \qquad x = 1, 2, ..., n$$

This is the zero truncated Poisson Distribution.

Moments The r^{th} moment about origin of zero truncated Poisson distribution is obtained as,

$$\mu_r' = E(X^r) = \sum_{x=1}^{n} \ x^r \ p_1(x) = \sum_{x=0}^{n} \ x^r \ \frac{\lambda^x \ e^{-\lambda}}{x! \ (1-e^{-\lambda})},$$

$$= \frac{1}{(1-e^{-\lambda})} \quad \sum_{x=0}^{n} \ x^r \ \frac{\lambda^x \ e^{-\lambda}}{x!},$$

$$= \frac{1}{(1-e^{-\lambda})} \ \times \ (r^{th} \text{ moment about origin of Binomial Distribution.})$$

$$\text{Mean} = \mu_1' = E(X) = \frac{\lambda}{(1-e^{-\lambda})}$$

$$\mu_2' = E(X^2) = \frac{\lambda^2 + \lambda}{(1 - e^{-\lambda})}$$

$$\text{Variance } \mu_2 = \mu_2' - (\mu_1')^2 = \frac{\lambda^2 + \lambda}{(1 - e^{-\lambda})} - \frac{\lambda^2}{(1 - e^{-\lambda})^2}$$

$$\mu_2 = \frac{\lambda}{(1 - e^{-\lambda})} \left[1 + \lambda - \frac{\lambda}{1 - e^{-\lambda}}\right]$$

Similarly, we can find μ_3' and μ_4'. Also β_1 and β_2.

References

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