Bayes Theorem

Theorem If $B_1, B_2, ..., B_n$ is a collection of mutually disjoint events and $P(B_j) \neq 0$, $(j = 1, 2, ..., n)$ then for every event C which is a subset of $\cup_{j=1}^{n} B_j$ such that $P(C) > 0$, we have

$$P(B_j|C) = \frac{P(B_j)P(C|B_j)}{\sum_{j=1}^{n} P(B_j)P(C|B_j)}, \quad j = 1, 2, ...n$$

Proof: Since $C \subset \cup_{j=1}^{n} B_j$, By using the law of total probability, we have

$$P(C) = \sum_{j=1}^{n} P(B_j)P(C|B_j)$$

By the conditional probability $P(C \cap B_j) = P(B_j)P(C|B_j)$ and also

$$P(B_j|C) = \frac{P(C \cap B_j)}{P(C)} = \frac{P(B_j)P(C|B_j)}{\sum_{j=1}^{n} P(B_j)P(C|B_j)}$$

Ex. There are two bags I and II. Bag I contains 3 white and 4 black balls and Bag II contains 5 white and 6 black balls. If one ball is selected at random from one of the bags and is found to be white. Find the probability that it was drawn from bag I?

Sol. Let E_1 and E_2 be the events of selecting bag I and bag II respectively.

Then $P(E_1) = \frac{1}{2}$ and $P(E_2) = \frac{1}{2}$.

Let W be the event of drawing a white ball and B be the event of drawing a black ball.

Then $P(W|E_1) =$ Probability of selecting a white ball from Bag $I = \frac{3}{7}$

Then $P(W|E_2) =$ Probability of selecting a white ball from Bag $II = \frac{5}{11}$

By Baye's Theorem, $P(E_1|W) = \dfrac{P(E_1).P(W|E_1)}{P(E_1).P(W|E_1) + P(E_2).P(W|E_2)}$

$$= \frac{(1/2) * (3/7)}{(1/2) * (3/7) + (1/2) * (5/11)}$$

$$= \frac{(3/7)}{(3/7) + (5/11)} = \frac{33}{68}.$$

Ex. Two urns I and II contain respectively 3 white and 2 black bails, 2 white and 4 black balls. One ball is transferred from urn I to urn II and then one is drawn from the latter. It happens to be white. What is the probability that the transferred ball was white.

Sol. Define, B_1 - Transfer a white ball from Urn I to Urn II,

B_2 - Transfer a black ball from Urn I to Urn II.,

A- Select a white ball from Urn II.

Here $P(B_1) = 3/5$, $P(B_2) = 2/5$, $P(A|B_1) = 3/7$, $P(A|B_2) = 2/7$ and $P(B_1|A) =?$.

By Baye's Theorem, $P(B_1|A) = \dfrac{P(B_1).P(A|B_1)}{P(B_1).P(A|B_1) + P(B_2).P(A|B_2)}$

$$= \frac{(3/5) * (3/7)}{(3/5) * (3/7) + (2/5) * (2/7)} = 9/13.$$

Ex. There are three bags $I,\ II$ and III. Bag I contains 4 white and 6 black balls , Bag II contains 6 white and 2 black balls and Bag III contains 5 white and 1 black balls. If a bag is selected at random and one ball is drawn and that found ball to be black. Find the probability that it was drawn from bag I?

Sol. Let $B_1,\ B_2$ and B_3 be the events of selecting bag $I,$ bag II and bag III respectively. Then $P(B_1) = \frac{1}{3},\ P(B_2) = \frac{1}{3}$ and $P(B_3) = \frac{1}{3}$.

Let C be the event of drawing a black ball. Then

$P(C|B_1) =$ Probability of selecting a Black ball from Bag $I = \frac{6}{10} = \frac{3}{5}$

$P(C|B_2) =$ Probability of selecting a Black ball from Bag $II = \frac{2}{8} = \frac{1}{4}$

$P(C|B_3) =$ Probability of selecting a Black ball from Bag $III = \frac{1}{6}$

By Baye's Theorem,

$$P(B_1|C) = \frac{P(B_1).P(C|B_1)}{P(B_1).P(C|B_1) + P(B_2).P(C|B_2) + +P(B_3).P(C|B_3)}$$

$$= \frac{(1/3) * (3/5)}{(1/3) * (3/5) + (1/3) * (1/4) + (1/3) * (1/6)}$$

$$= \frac{(3/5)}{(3/5) + (1/4) + (1/6)} = \frac{36}{61}.$$

Ex. There are three bags $I,\ II$ and III. Bag I contains 1 white, 3 red and 2 black balls , Bag II contains 2 white, 1 red and 1 black balls and Bag III contains 4 white, 3 red and 5 black balls. If a bag is selected at random and two balls are drawn and that found both balls to be white and red. Find the probability that these both were drawn from bag I?

Sol. Let $B_1,\ B_2$ and B_3 be the events of selecting bag $I,$ bag II and bag III respectively.

Then $P(B_1) = \frac{1}{3},\ P(B_2) = \frac{1}{3}$ and $P(B_3) = \frac{1}{3}$.

Let C be the event of drawing a white and a red ball. Then

$P(C|B_1) =$ Probability of selecting a white and a red ball from Bag $I = \frac{{}^1C_1 \times^3 C_1}{{}^6C_2} = \frac{1}{5}$

$P(C|B_2) =$ Probability of selecting a Black ball from Bag $II = \frac{{}^2C_1 \times^1 C_1}{{}^4C_2} = \frac{1}{3}$

$P(C|B_3) =$ Probability of selecting a Black ball from Bag $III = \frac{{}^4C_1 \times^3 C_1}{{}^12C_2} = \frac{2}{11}$

By Baye's Theorem,

$$P(B_1|C) = \frac{P(B_1).P(C|B_1)}{P(B_1).P(C|B_1) + P(B_2).P(C|B_2) + +P(B_3).P(C|B_3)}$$

$$= \frac{(1/3) * (1/5)}{(1/3) * (1/5) + (1/3) * (1/3) + (1/3) * (2/11)}$$

$$= \frac{(1/5)}{(1/5) + (1/3) + (2/11)} = \frac{33}{118}.$$

Assignment

Q. Bags I and II each contain 2 white and 2 black balls. 1 ball is selected from bag I and transferred to bag II. 1 Ball is drawn from bag II and found to be black. What is the probability that the transferred black was black.

Q. There are three bags I, II and III. Bag I contains 3 white and 5 black balls , Bag II contains 5 white and 1 black balls and Bag III contains 4 white and 2 black balls. If a bag is selected at random and one ball is drawn and that found ball to be white. Find the probability that it was drawn from bag II?

Q. There are two bags I and II. Bag I contains 4 red and 5 black balls and Bag II contains 6 red and 7 black balls. If one ball is selected at random from one of the bags and is found to be red. Find the probability that it was drawn from bag II?

References

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