Radiation From an Accelerated Charged Particle

Course: MPHYCC-06 Electrodynamics and Plasma Physics

(M.Sc. Sem-II)

Dr. Sanjay Kumar

Assistant Professor

Department of Physics

Patna University

Contact Details: Email- sainisanjay35@gmail.com;

Contact No.- 9413674416

Power Radiated by an Accelerated Point Charge

To calculate the power radiated by a point charge moving with an arbitrary velocity **v** (hence, having non-zero acceleration **a**), from previous lectures, we recall the expressions of electric **(E)** and magnetic **(B)** field generated by the point charge moving on a trajectory **w**(t) as shown in figure 1:

$$\boxed{\mathbf{E}(\mathbf{r},t) = \frac{q}{4\pi\epsilon_0}\frac{\imath}{(\boldsymbol{\imath}\cdot\mathbf{u})^3}[(c^2-v^2)\mathbf{u} + \boldsymbol{\imath}\times(\mathbf{u}\times\mathbf{a})].} \quad \text{---- (1)}$$

$$\boxed{\mathbf{B}(\mathbf{r},t) = \frac{1}{c}\hat{\boldsymbol{\imath}}\times\mathbf{E}(\mathbf{r},t).} \quad \text{----- (2)}$$

where

$$\mathbf{u} \equiv c\hat{\boldsymbol{\imath}} - \mathbf{v}$$

$$\boldsymbol{\imath} = \mathbf{r} - \mathbf{w}(t_r)$$

and retarded time $t_r = t - |\mathbf{r}-\mathbf{w}(t_r)|/c = t - \imath/c$

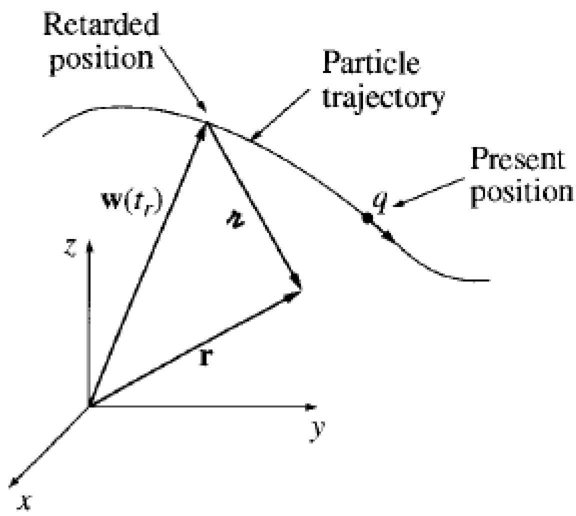

Figure 1

As mentioned in the previous lecture note, the first term in equation (1) is known as “Coulomb field” or “velocity field” while the second term is called as “radiation field” or “acceleration field”.

To calculate the power radiated by the point charge, first of all we need to find an expression for the Poynting vector:

$$\boldsymbol{S} = \frac{1}{\mu_0}(\boldsymbol{E} \times \boldsymbol{B})$$

$$\boldsymbol{S} = \frac{1}{\mu_0 c}[\boldsymbol{E} \times (\hat{\boldsymbol{\imath}} \times \boldsymbol{E})] \qquad \text{(using equation (2))}$$

$$\boldsymbol{S} = \frac{1}{\mu_0 c}[E^2 \hat{\boldsymbol{\imath}} - \boldsymbol{E}(\boldsymbol{E} \cdot \hat{\boldsymbol{\imath}})] \qquad \text{------ (3)}$$

From the discussions of the previous lecture, we recall that the second term of electric field (called as radiation field and was denoted by $\mathbf{E_r}$) in equation (1) is proportional to $(1/\boldsymbol{\imath})$ and, therefore, E_r^2 is proportional to $(1/\boldsymbol{\imath}^2)$. Because of this, E_r^2 doesn't vanish at large distances and contribute to the radiation. In contrast, the first term of electric field $\mathbf{E_c}$ (coulomb or velocity field) has the dependency of $(1/\boldsymbol{\imath}^2)$ and therefore E_c^2 vanishes at large distances. Because of these arguments, in equation (3) we only need to consider the radiation field:

$$\boldsymbol{E_r} = \frac{q\boldsymbol{\imath}}{4\pi\varepsilon_0(\boldsymbol{\imath} \cdot \boldsymbol{u})^3}(\boldsymbol{\imath} \times \boldsymbol{u} \times \boldsymbol{a}) \qquad \text{------ (4)}$$

So, now from equation (3), the Poynting flux responsible for the radiation can be given as:

$$\boldsymbol{S_r} = \frac{1}{\mu_0 c}[E_r^2 \hat{\boldsymbol{\imath}} - \boldsymbol{E_r}(\boldsymbol{E_r} \cdot \hat{\boldsymbol{\imath}})] \qquad \text{------ (5)}$$

From equation (4), we note that the $\mathbf{E_r}$ is perpendicular to $\boldsymbol{\imath}$ because the expression of $\mathbf{E_r}$ involves a vector triple product of vectors $\boldsymbol{\imath}$, **u**, and **a**. Hence, $\mathbf{E_r}$ and $\boldsymbol{\imath}$ normal to each other and, hence equation (5) modifies as:

$$\boldsymbol{S_r} = \frac{1}{\mu_0 c}(E_r^2 \hat{\boldsymbol{\imath}}) \qquad \text{------ (6)}$$

From equation (4), we can rewrite the expression of $\mathbf{E_r}$ as:

$$\boldsymbol{E_r} = \frac{q\boldsymbol{\imath}}{4\pi\varepsilon_0(\boldsymbol{\imath} \cdot \boldsymbol{u})^3}[\boldsymbol{u}(\boldsymbol{\imath} \cdot \boldsymbol{a}) - \boldsymbol{a}(\boldsymbol{\imath} \cdot \boldsymbol{u})] \qquad \text{------ (7)}$$

Now, under the non-relativistic limit v<<c, $\mathbf{u}=c\hat{\mathscr{r}}$ **- v** can be approximated as $\mathbf{u}=c\hat{\mathscr{r}}$. Then, $\boldsymbol{\mathscr{r}}.\mathbf{u}=\mathscr{r}\,c$ and using this in equation (7), we get

$$\boldsymbol{E}_r=\frac{q}{4\pi\varepsilon_0\mathscr{r}^2c^3}[\hat{\mathscr{r}}c(\boldsymbol{\mathscr{r}}.\boldsymbol{a})-\boldsymbol{a}(\mathscr{r}c)]$$

$$\boldsymbol{E}_r=\frac{q}{4\pi\varepsilon_0\mathscr{r}c^2}[\hat{\mathscr{r}}(\hat{\mathscr{r}}.\boldsymbol{a})-\boldsymbol{a}]$$

$$\boldsymbol{E}_r=\frac{q\mu_0}{4\pi\mathscr{r}}[\hat{\mathscr{r}}(\hat{\mathscr{r}}.\boldsymbol{a})-\boldsymbol{a}] \qquad \text{------ (8)}$$

Use the expression of $\mathbf{E_r}$ from equation (8) into equation (6) to calculate the Poynting flux:

$$\boldsymbol{S}_r=\frac{1}{\mu_0 c}\left(\frac{q\mu_0}{4\pi\mathscr{r}}\right)^2[\hat{\mathscr{r}}(\hat{\mathscr{r}}.\boldsymbol{a})-\boldsymbol{a}]^2\hat{\mathscr{r}}$$

$$\boldsymbol{S}_r=\frac{\mu_0 q^2}{16\pi^2\mathscr{r}^2c}[(\hat{\mathscr{r}}.\boldsymbol{a})^2+a^2-2(\hat{\mathscr{r}}.\boldsymbol{a})(\hat{\mathscr{r}}.\boldsymbol{a})]\hat{\mathscr{r}}$$

$$\boldsymbol{S}_r=\frac{\mu_0 q^2}{16\pi^2\mathscr{r}^2c}[a^2-(\hat{\mathscr{r}}.\boldsymbol{a})^2]\hat{\mathscr{r}}$$

If the angle between $\mathscr{r}$ and **a** is θ then

$$\boldsymbol{S}_r=\frac{\mu_0 q^2a^2}{16\pi^2c}[1-\cos^2\theta]\hat{\mathscr{r}}=\frac{\mu_0 q^2a^2}{16\pi^2c}\left(\frac{\sin^2\theta}{\mathscr{r}^2}\right)\hat{\mathscr{r}} \qquad \text{----- (9)}$$

To calculate the power, from the previous lecture, we know that the power radiated by the point charge is given by

$$P=\oint\boldsymbol{S}_r.d\boldsymbol{a} \qquad \text{------ (10)}$$

To calculate the above integral or the power radiated by the charge particle at the retarded time t_r, we draw a sphere of large radius $\mathscr{r}$ (as shown in figure 2) and consider that the particle is situated at the origin of the sphere at time t_r. The power radiated by the particle travels with the speed of light (c) and takes $\mathscr{r}$/c time to reach the surface of the sphere and, at this moment, we integrate the Poynting flux over the surface for calculating the power (as documented in equation (10)).

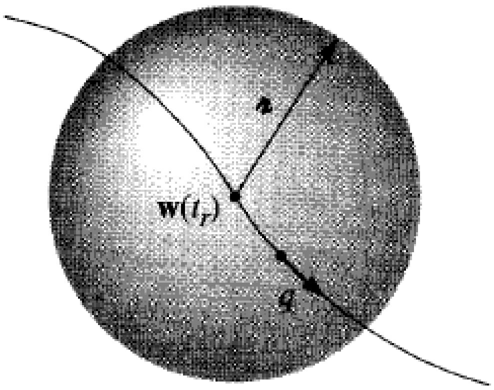

Figure 2

Now using the expression of $\mathbf{S_r}$ from equation (9) and area element for a spherical surface **da =** $\mathscr{r}^2 \sin\theta\, d\theta d\Phi\, \hat{\mathscr{r}}$ in equation (10) we get:

$$P = \iint \frac{\mu_0 q^2 a^2}{16\pi^2 c}\left(\frac{\sin^2\theta}{\mathscr{r}^2}\right)\mathscr{r}^2 \sin\theta\, d\theta\, d\Phi \qquad \text{---------- (11)}$$

$$P = \frac{\mu_0 q^2 a^2}{16\pi^2 c}\iint \sin^3\theta\, d\theta\, d\Phi$$

$$P = \frac{\mu_0 q^2 a^2}{6\pi c} \qquad \text{----------- (12)}$$

This is the expression of power radiated by a charged particle moving with a low velocity (v<<c). *This is also called the Larmor formula.* Please note that the power only depends on the charge of the particle and the acceleration of the particle. With an increase in acceleration of a charged particle, the power radiated by the particle also enhances.

Angular distributation of the radiation:

To understand the angular distributation of the power radiated by a charged particle, we again consider equation (11):

$$P=\iint \frac{\mu_0 q^2 a^2}{16\pi^2 c}\left(\sin^2\theta\right)\sin\theta\, d\theta\, d\Phi \qquad \text{------ (13)}$$

We know that differential solid angle element dΩ = sinθdθdΦ (see figure 3),

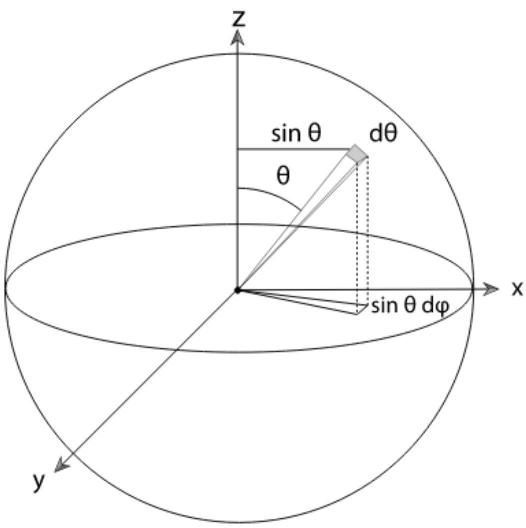

Figure 3

Then, equation (13) can be rewritten as:

$$P=\iint \frac{\mu_0 q^2 a^2}{16\pi^2 c}\left(\sin^2\theta\right)d\Omega$$

$$\frac{dP}{d\Omega}(\theta)=\frac{\mu_0 q^2 a^2}{16\pi^2 c}\left(\sin^2\theta\right) \qquad \text{------ (14)}$$

The above equation determines the power radiated in per unit solid angle and, hence, represents the angular distributation of the power radiated by the charged particle. Note that power radiated in the forward and backward direction of the motion of the particle is zero because for the forward direction $\theta = 0^0$ and the backward direction $\theta = 180^0$.

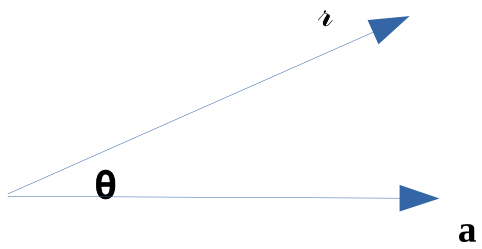

Figure 4

Maximum power is radiated in the perpendicular direction of the motion of the charged particle because $\theta = 90^0$. In three-dimension (3D), most of the power is radiated in the donut shape (see figure 5) about the direction of motion of the charged particle (or the instantaneous direction of acceleration).

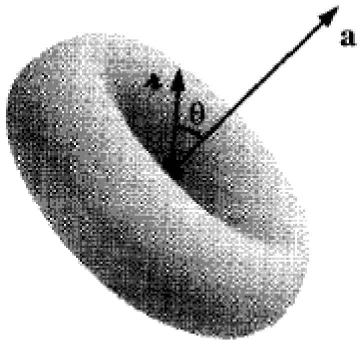

Figure 5

Referece:

"Introduction to Electrodynamics" by David J. Grifitths.

Thanks for the attention!