Eigenvalues and Eigenvectors: Determinant Method

MPHYCC-05 Unit-IV Semester-II

Eigenvalues and Eigenvectors:

Eigen values and corresponding Eigen values are having special importance in the linear algebra. Eigen is a German word which has meaning as proper or characteristic. It is initially applied to study the principal axes of rotational motion of rigid bodies. Moreover, Eigenvectors and Eigenvalues also utilize to understand the related to the atomic orbitals, vibrational analysis, facial recognition, matrix diagonalization and so on.

Eigen vector of a linear transformation is a non-zero vector. The vector changes by a scalar factor when the linear transform applied on it and the scalar factor is known as the corresponding Eigen value vector. Formal definition of Eigenvectors and Eigenvalues is as follows.

Let us assume that T is a linear transformation from a vector space V over a field F into itself. And v is a nonzero vector in V, then v is an eigenvector of T if T(v) (transformation T, acts on the vector v) is a scalar multiple of v. This can be mathematically written as:

$$T(v) = \lambda\, v$$

Where λ is a scalar in F, known as the characteristic value eigenvalue associated with v. However, for the finite dimensional vector, the above can be written in terms of matrix. And the corresponding matrix equivalent is:

$$A\,w = \lambda\, w$$

Where A is matrix representation of a linear transform, T and w is the coordinate vector of v. In principle, the linear transform can take the many form so the eigenvector. The linear transformation could take the form of n by n matrix, and then eigenvector is represented by n by 1 matrix. The set of eigenvectors of the linear transformer is known as the eigensystem of that transformation. Other example, the linear transform could be a differential operator such as $\frac{d}{dx}$ and in this case the eigenvectors are function known as eigenfunction such as $e^{\lambda x}$ which are scaled by the operator.

$$\frac{d}{dx}e^{\lambda x} = \lambda e^{\lambda x}$$

Let us assume the linear transform of n-dimensional vector, v by an n by n matrix in such a way that:

$$\mathrm{A\,v} = \lambda\,\mathrm{v}$$

Then v is the eigenvector of the linear transformation A and the scalar multiplier λ is the corresponding eigenvalue. The above equation can be written as:

$$(\mathrm{A} - \lambda\mathrm{I})\mathrm{v} = 0$$

And I is the n by n identity matrix and 0 is the zero vector. The above equation known as eigenvalue equation has a solution for a nonzero v if and only if the determinant of the matrix $(\mathrm{A} - \lambda\mathrm{I})$ is zero; that is

$$|A - \lambda I| = 0$$

The above equation is a polynomial function of the variable λ and the degree of this polynomial is n that is order of the matrix A. This polynomial is known as characteristic polynomial of A and equation is called as characteristic equation of A. The equation can be written as the product of n factored term as:

$$|A - \lambda I| = (\lambda - \lambda_1)(\lambda - \lambda_2) \ldots\ldots(\lambda - \lambda_n)$$

Whereλ_i's are the roots of the polynomial and it can be real but in general it is complex number. λ_i's that are λ_1, λ_2, …,λ_n which may or may not be distinct values. The properties of the matrix and its eigenvalues are summarized below.

Let us assume that A is a n by n matrix with eigenvaluesλ_1, λ_2, …,λ_n then

- The trace of A, tr(A) which is defined as the sum of the its diagonal elements is also equal to the sum of its all eigenvalues as:

  $tr(A) = \sum_{i=1}^{n} a_{ii} = a_{11} + a_{22} + \cdots + a_{nn}$

$$= \sum_{i=1}^{n} \lambda_i = \lambda_1 + \lambda_2 + \cdots + \lambda_n$$

- The determinant of A, det(A) products of its all eigenvalues as: $det(A) = \prod_{i=1}^{n} \lambda_i = \lambda_1 * \lambda_2 * \ldots * \lambda_n$

- Eigenvalues of A^m that is the eigenvalues of m^{th} power of A, for any positive integer m; are $\lambda_1^m, \lambda_2^m, \ldots, \lambda_n^m$

- The matrix A is invertible iff every eigenvalues is non-zero. If A is invertible, then eigenvalues of A^{-1} are; $\frac{1}{\lambda_1}, \frac{1}{\lambda_2}, \ldots, \frac{1}{\lambda_n}$ because the characteristic polynomial of inverse matrix is the reciprocal polynomial of the original.

- Eigenvalues of A+I, where I is the identity matrix of same dimension as A are: $\lambda_1 + 1, \lambda_2 + 1, \ldots, \lambda_n + 1$. Moreover the eigenvalues of A+βI where β is complex number are: $\lambda_1 + \beta, \lambda_2 + \beta, \ldots, \lambda_n + \beta$.

- If A is Hermitian that is complex conjugate transpose $(A^*)^T$ is equal to A, then its every eigenvalue is real.

- If A unitary that is complex conjugate transpose $(A^*)^T$ is equal to A itself: $A (A^*)^T = (A^*)^T A = I$. then every eigenvalue has absolute value that is $|\lambda_i| = 1$

- Eigenvalues of a diagonal matrix are the diagonal elements themselves. Moreover, eigenvalues of the triangular matrix are also the elements of the main diagonal similar to the diagonal matrix.

- The eigenvalue of an invertible matrix (non-singular) is non-zero but zero is an eigenvalue of a singular matrix. Suppose matrix A is not singular then we can shift A by a multiple of I to make it singular.

Techniques for Eigenvalues and Eigenvector:

There are various methods are reported in the literature to find the eigenvalues and eigenvectors of a matrix. However, we discussed only characteristic equation solving method, power method, inverse power method and Jacobi method.

Determinant Method: Characteristic Equation

Eigenvalues and eigenvectors are having special importance in the linear algebra. Eigen is a German word which has meaning as proper or characteristic. It is initially applied to study the principal axes of rotational motion of rigid bodies. Below are the simple steps to find the eigenvalues and eigenvectors of a matrix. Write the eigenvalue equation (A-λI)x=0

- Compute the determinant of $|A - \lambda I|$ that is a characteristic polynomial of λ of degree n.

- Find the roots of this polynomial: the polynomial has n roots and these roots are the eigenvalues of the matrix.

- Substitute the eigenvalues into the eigenvalue equation, one by one, then calculate the eigenvector

- Thus the eigenvector, x, for each eigenvalue, λ, can be calculated by solving the equation (A-λI)x=0

- Sometimes for the larger matrix it is not easy to solve the equation (A-λI)x=0. In that case one can apply row reduce method to the (A-λI) matrix. Then replace the (A-λI) by the resulting row-reduced matrix and solve (row-reduced matrix)x=0. The calculation leads to the required eigenvector.

Example

Example is given below to have a better understanding about the method to calculate the eigenvalue and eigenvector.

Example 1: Find the eigenvectors and eigenvalues of the following matrix.

$$A=\begin{bmatrix} 8 & 3 \\ 2 & 7 \end{bmatrix}$$

Solution: Characteristic equation: $|A-\lambda I|=0$

$$|A-\lambda I|=\begin{vmatrix} 8-\lambda & 3 \\ 2 & 7-\lambda \end{vmatrix}=0 \qquad I=\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$\lambda^2-15\lambda+50=0$$

$(\lambda-10)(\lambda-5)=0$ gives $\lambda=10$ and $\lambda=5$

Thus the eigenvalues are 10 and 5. Let the Corresponding eigenvectors are x_1 and x_2. Then For $\lambda=10$;

$$(A-\lambda I)\,x_1=\begin{bmatrix} -2 & 3 \\ 2 & -3 \end{bmatrix}\begin{bmatrix} x_1^1 \\ x_1^2 \end{bmatrix}=0$$

$\begin{bmatrix} -2\,x_1^1+3x_1^2 \\ 2\,x_1^1-3x_1^2 \end{bmatrix}=0$ This equation gives;

$-2\,x_1^1+3x_1^2=0$ and $2\,x_1^1-3x_1^2=0$

let $x_1^1=1$; then $x_1^2=2/3$; we can simplify this in integral form as: $x_1^1=3$, $x_1^2=2$. Therefore

$$\text{Eigenvector } x_1=\begin{bmatrix} 3 \\ 2 \end{bmatrix}$$

Then For $\lambda=5$; $(A-\lambda I)\,x_2=\begin{bmatrix} 3 & 3 \\ 2 & 2 \end{bmatrix}\begin{bmatrix} x_2^1 \\ x_2^2 \end{bmatrix}=0$

$\begin{bmatrix} 3\,x_2^1+3x_2^2 \\ 2\,x_2^1+2x_2^2 \end{bmatrix}=0$ This equation gives;

$3x_2^1+3x_2^2=0$ and $2x_2^1+2x_2^2=0$

let $x_2^1=1$; then $x_2^2=-1$; Therefore

$$\text{Eigenvector } x_2=\begin{bmatrix} 1 \\ -1 \end{bmatrix}$$

It is also clear that

$A\,x_1=10\,x_1$

$A\,x_2=5\,x_2$

Python scripts and outputs

We can write a simple python code to find the eigenvector and eigenvalues of a square matrix using ***numpy***. However, the eigenvector resulting from the code is not in an updated form, we should simplify it for further use. The python code is as following where eigenvectors are the columns of the output eigenvector matrix.

Script:

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```
import numpy as np
row = int(input("Enter the number of rows:"))
column = int(input("Enter the number of columns:"))
print("Enter the matrix element separated by space: ")
if(row == column):

# User input of entries in a
# single line separated by space
    entries = list(map(int, input().split()))

# For printing the matrix
    matrixm = np.array(entries).reshape(row, column)

# Getting Eigenvalues and Eigenvector: w & v
    w, v = np.linalg.eig(matrixm)
    print('The matrix:', matrixm,)
    print('Eigenvalues are:', w)
    print('Eigenvectors are:', v)
else:
   print("Matrix should be square")
```

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Outputs:

After completing the code for various matrices we have the following outputs:

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```
Enter the number of rows:2
Enter the number of columns:2
Enter the matrix element separated by space:
0 1 -6 5
The matrix: [[ 0  1]
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[-6 5]]
Eigenvalues are: [2. 3.]
Eigenvectors are: [[-0.4472136 -0.31622777]
[-0.89442719 -0.9486833]]

Thus the eigenvalues are: 2 & 3

And corresponding eigenvectors are: [-0.4472136, -0.89442719] that is [1, 2] and [-0.31622777, -0.9486833] that is [1, 3]. Therefore eigenvectors are: [1, 2] & [1, 3]

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Enter the number of rows:2
Enter the number of columns:2
Enter the matrix element separated by space:
8 3 2 7
The matrix: [[8 3]
[2 7]]
Eigenvalues are: [10. 5.]
Eigenvectors: [[0.83205029 -0.70710678]
[0.5547002   0.70710678]]

Thus the eigenvalues are: 10 & 5

And corresponding eigenvectors are: [0.83205029, 0.5547002] that is [3, 2] and [0.70710678, -0.70710678] that is [1, -1]. Therefore eigenvectors are: [3, 2] & [1, -1]

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Enter the number of rows:3
Enter the number of columns:3
Enter the matrix element separated by space:
3 2 3 0 6 10 0 0 2
The matrix: [[3 2 3]
[0 6 10]
[0 0 2]]
Eigenvalues are: [3. 6. 2.]
Eigenvectors: [[1.   0.5547002   0.59628479]
[0.   0.83205029 -0.74535599]
[0.   0.   0.2981424]]

Thus the eigenvalues are: 3, 2 & 3

And corresponding eigenvectors are: [1, 0, 0], [0.5547002, 0.83205029, 0] that is [2, 3, 0] and [0.59628479, -0.74535599, 0.2981424] that is [4, -5, 2]. Therefore eigenvectors are: [1, 0, 0], [2, 3, 0] & [4, -5, 2]

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