Quantum Field Theory
M.Sc. 4th Semester
MPHYEC-1: Advanced Quantum Mechanics
Unit III (Part 6)
Topic: System of Fermions

Compiled b

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System of Fermions

For a system of fermions, the number of particles n_k in any state should be restricted to 0 and 1, to be in accordance with Pauli's exclusion principle. Jordon and Wigner have shown that this condition could be realised by replacing the above commutation relations by the following anticommutation relations:

$$[a_k, a_l^\dagger] = \delta_{kl} \text{ and } [a_k, a_l]_+ = [a_k^\dagger, a_l^\dagger]_+ = 0 \qquad \ldots.(1)$$

From equation (1), we have

$$a_k a_k^\dagger + a_k^\dagger a_k = 1 \text{ and } a_k a_k = a_k^\dagger a_k^\dagger = 0 \qquad \ldots (2)$$

Again, we define the particle number operator in the k^{th} state N_k by

$$N_k = a_k a_k^\dagger \qquad \ldots (3)$$

Each N_k commutes with all the others and therefore they can be diagonalize simultaneously. The eigenvalue of $N_k$can be obtained by evaluating the square of ${N_k}^2$,

$${N_k}^2 = a_k a_k^\dagger a_k a_k^\dagger = a_k^\dagger (a_k a_k^\dagger) a_k = a_k^\dagger (1 - a_k^\dagger a_k) a_k$$

$$= a_k^\dagger a_k = N_k \qquad \ldots (4)$$

Since the second term is zero by virtue of equation (2). N_k is diagonalized with eigen value n_k and therefore , ${N_k}^2$ would also be diagonalized with eigenvalue ${n_k}^2$. Hence equation (4) is equivalent to,

$${n_k}^2 = n_k \text{ or } {n_k}^2 - n_k = 0 \text{ or } n_k(n_k - 1) = 0 \qquad \ldots (5)$$

Which gives, $\quad n_k = 0,1 \qquad \ldots (6)$

Thus, the eigenvalue of N_k are 0 and 1. As in the case of bosons, we can define a number operator N representing the total number of particles by,

$$N = \sum_k N_k \qquad \ldots (7)$$

The eigenvalues of N are the positive integer including zero as before.

The expression

$$H = \sum_k \sum_l {a_k}^\dagger a_l \int \left(\frac{\hbar^2}{2m} . \nabla u_k^* \nabla u_l + V u_k^* u_l \right) d^3 r \qquad \ldots (8)$$

(see https://www.patnauniversity.ac.in/e-content/science/physics/MScPhy24.pdf) which was valid for bosons is also valid for fermions. Again, the energy eigenvalue is given by equation:

$$\int \nabla u_k^* \, \nabla u_l d^3r = \int u_k^* \nabla u_l ds - \int u_k^* \nabla^2 u_l \, d^3r \qquad \dots (9)$$

The following relations also result from the anticommutator rules in equation (1),

$$a_k|n_1, n_2, \dots, n_k >= \ (-1)^{s_k} n_k |n_1, n_2, \dots, n_{k-1,\dots} > \qquad \dots(10)$$

$$a_k^\dagger|n_1, n_2, \dots, n_k >= \ (-1)^{s_k}(1 - n_k)|n_1, n_2, \dots, n_{k+1,\dots} > \qquad \dots(11)$$

$$|n_1, n_2, \dots, n_k >= \ (a_1^\dagger)^{n_1}(a_2^\dagger)^{n_2} \dots \left(a_k^\dagger\right)^{n_k} \dots |0 > \qquad \dots (12)$$

Where, $S_k = \sum_{r=1}^{k-1} n_r$ … (13)

A representation for the operators a_k and $a_k^\dagger$ can be obtained if the system has only one state. The number operator N_k has the eigenvalues 0 and 1. Hence N_k can be represented by the diagonal matrix

$$N_k = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \qquad \dots (14)$$

Matrices for a_k and $a_k^\dagger$ satisfy the condition

$$a_k a_k = a_k^\dagger a_k^\dagger = 0 \text{ are}$$

$$a_k = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \qquad \text{and } a_k^\dagger = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \qquad \dots (15)$$

The kets representing the eigenvalues zero and one for the operator a_k can be expressed as:

$$|0> = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \quad \text{and } |1> = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

In real situation, the number of states of the system is infinite and not single as assumed. Hence, explicit simple matrix like the preceding one is not possible.

<u>Reference:</u>

1. An Introduction to Quantum Field Theory by Mrinal Dasgupta
2. QUANTUM FIELD THEORY *A Modern Introduction* by Michio Kaku
3. First Book of Quantum Field Theory by Amitabha Lahiri & P. B. Pal
4. Quantum mechanics by G.S. Chaddha