Classification of 2nd Linear Partial Differential Equations: Poisson's, Wave and Diffusion Equation

Course: MPHYCC-05 Modeling and Simulation
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Linear Partial Differential Equations (PDEs)

A partial differential equation (PDE) can be defined as an equation which have an unknown function/variable of more than one independent known variable and at least one partial derivative of the function. To further clarify, we consider a few examples: Lets consider a function *f=f(x,y)* where *x* and *y* represent independent variables, then

$$\frac{\partial f}{\partial x}+f=0 \qquad \text{------ (1)}$$

$$\frac{\partial f}{\partial x}+\frac{\partial f}{\partial y}=0 \qquad \text{------- (2)}$$

$$\frac{\partial f}{\partial x}+\frac{\partial f}{\partial y}+f=0 \qquad \text{------- (3)}$$

$$\frac{\partial^2 f}{\partial x^2}+\frac{\partial^2 f}{\partial y^2}+\frac{\partial^2 f}{\partial x \partial y}+f=0 \qquad \text{------- (4)}$$

$$3\frac{\partial^3 f}{\partial x^3}+2\frac{\partial^2 f}{\partial y^2}+3\frac{\partial^2 f}{\partial x \partial y}+f=0 \qquad \text{------- (5)}$$

$$(xy)\frac{\partial^2 f}{\partial x^2}+(x^2+y^2)\frac{\partial^2 f}{\partial y^2}+y\frac{\partial^2 f}{\partial x \partial y}+f=0 \qquad \text{-------- (6)}$$

$$(xf)\frac{\partial^2 f}{\partial x^2}+(x^2+f^2)\frac{\partial^2 f}{\partial y^2}+y\frac{\partial^2 f}{\partial x \partial y}+f^2=0 \qquad \text{-------- (7)}$$

equations (1)-(7) represent the PDEs. To proceed further, it is important to mention that, in PDEs, it is common to denote partial derivatives using subscripts. That is:

$$f_x=\frac{\partial f}{\partial x} \quad ; \quad f_{xx}=\frac{\partial^2 f}{\partial x^2}$$

$$f_y=\frac{\partial f}{\partial y} \quad ; \quad f_{yy}=\frac{\partial^2 f}{\partial y^2}$$

$$f_{xy}=\frac{\partial^2 f}{\partial x \partial y}$$

Order of PDE: The order of a PDE is the order of the highest order derivative that appears in the PDE. For example, equations (1)-(3) have order one. While equations (4), (6) and (7) are second order and equation (5) is third order PDE.

Linear PDE: If the dependent variable and all its partial derivatives appear linearly in any PDE then such an equation is called linear PDE otherwise a non-linear PDE. For example, equations (1)-(6) are linear as the multiplicative cofficients of the dependent/unknown variable *f* and all its derivatives are either constant or independent of the dependent variable *f*. These cofficients may depend on the independent variables *x* and *y*. For example in equation (6), the cofficient of f_{xx} is xy, cofficient of f_{yy} is $(x^2 + y^2)$, cofficient of f_{xy} is y and cofficient of f is 1. However, equation (7) represent a non-linear PDE because, in this equation, the cofficients depend on f.

Homogeneous PDE: If all the terms of a PDE contains the dependent variable or its partial derivatives then such a PDE is called homogeneous partial differential equation. All the equations (1)-(7) are homogeneous PDEs.

2nd Order Linear PDEs

We know that the various phenomena in physical systems are described by the second order linear PDEs. For example, wave propagation is described by the wave equation which is a second order PDE. Heat conduction in any medium is explained by the heat equation that also is a second order PDE. We also know that Laplace’s equation describes the steady physical state of the wave and heat conduction phenomena. Also the Possoin’s equation is used to determine the electrostatic field for a given charge distributation. It then becomes imperative to study the properties of the 2nd order linear PDEs. In the following, we will consider the general second order linear PDE and will reduce it to one of three distinct types of equations that have the wave, diffusion and Poisson’s equations as their canonical forms. Knowing the type of the equation allows one to use relevant methods for studying it, which are quite different depending on the type of the equation. One should compare this to the conic sections, which arise as different types of second order algebraic equations (quadrics). Since the hyperbola, given by the equation $x^2-y^2= 1$, has very different properties from the parabola $x^2 = y$, it is expected that the same holds true for the wave and diffusion equations as well.

The general second order linear PDE for an unknown function/variable *u=u(x,y)* can be given as:

$$A\frac{\partial^2 u}{\partial x^2}+B\frac{\partial^2 f}{\partial x\partial y}+C\frac{\partial^2 f}{\partial y^2}+D\frac{\partial u}{\partial x}+E\frac{\partial u}{\partial y}+Fu=G$$

or in short-notation:

$$Au_{xx}+Bu_{xy}+Cu_{yy}+Du_x+Eu_y+Fu=G \qquad \text{------- (8)}$$

where the coefficients A, B, C, D, F and G are in general functions of the independent variables *x, y,* but do not depend on the unknown function *u*. Notice that, the wave, diffusion and Poisson's equations all are second order PDEs with the constant cofficients, i.e. don't depend on *x, y.*

The classification of second order equations depends on the form of the first three terms (which contain the second order partial derivatives) of the equation (8). So, for simplicity of notation, we combine the last four terms (which contain first order partial derivatives and u and G) and rewrite the above equation in the following form:

$$Au_{xx}+Bu_{xy}+Cu_{yy}+H(u_x,u_y,u,G)=0 \qquad \text{------- (9)}$$

<u>Classification of 2nd Order Linear PDEs</u>

Classification of 2nd order linear PDEs as written in equation (9) depends on the sign of the quantity

$$\Delta = B^2 - 4AC \qquad \text{------ (10)}$$

which is called descriminant for equation (9). Note that A, B and C are multiplicative coefficients of u_{xx}, u_{xy} and u_{yy} respectively. As mentioned above, in general, A, B and C are the function of *x* and *y*. Therefore, the descriminant $\Delta = \Delta(x, y)$ is also function of *x* and *y*.

Then, the classification of second order linear PDEs is given in the following manner. At the point (x_0, y_0) the second order PDE (given by equation (9)) is called:

Hyperbolic	If $\Delta(x_o, y_o) > 0$
Parabolic	If $\Delta(x_o, y_o) = 0$
Elliptic	If $\Delta(x_o, y_o) < 0$

Notice that in general a second order equation may be of one type at a specific point, and of another type at some other point. This point is explicitly demonstrated in the notes by taking examples (see last page of the notes).

Types of the Poisson's, Wave and Diffusion Equations

Now, based on the classification discussed above, we will determine the type of the Poisson's equation, wave and diffusion equations.

Poisson's or Laplace's Equation: First of all, let us determine the type of the Poisson's or Laplace equation in two-dimensional (2D) space. We know that the Laplace's equation in 2D is given as:

$$\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}=0$$

or in short notations:

$$u_{xx}+u_{yy}=0 \qquad \text{----------- (11)}$$

Now, from a comparison of the Laplace equation (11) with equation (9) suggests that, for the Laplace equation (11), the coefficients are:

A=1, B=0 and C=1

Therefore, the descriminant for all the x and y is:

$$\Delta = B^2 - 4AC = 0-4=-4$$

As a result, for the Laplace equation the descriminant $\Delta < 0$. This means that the *Laplace equation is elliptic.*

Similarly, we know that the 2D Poisson's equation is given as:

$$\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}=h(x,y) \qquad \text{----------- (12)}$$

From a comparison of equation (12) and equation (9), for the Poisson's equation, we can determine A=1, B=0 and C=1. Hence, for the Poisson's equation also the descriminant $\Delta < 0$ and *the Poisson's equation is elliptic.*

Wave Equation: We know that the 1D wave equation for a wave moving with a velocity v is given as:

$$\frac{\partial^2 u}{\partial x^2}=\frac{1}{v^2}\frac{\partial^2 u}{\partial t^2}$$

or in short notations:

$$u_{xx} - \frac{1}{v^2} u_{tt} = 0 \qquad \text{--------- (13)}$$

Note that, in the wave equation the independent variables are *x* and *t* instead of *x* and *y*. Now, keeping this in mind, if we compare the wave equation (13) with equation (9) we can obtain:

$$A = 1, B = 0, C = -\frac{1}{v^2} \qquad \text{--------- (14)}$$

Then, the descriminant $\Delta = B^2 - 4\,AC = 0 + (1/v^2) = (1/v^2)$. Since speed *v* of a wave is always positive, therefore $(1/v^2) > 0$ *and* $\Delta > 0$. As a result, *the wave equation is always hyperbolic*.

Diffusion Equation: 1D diffusion equation is given as:

$$\frac{\partial u}{\partial t} = \lambda \frac{\partial^2 u}{\partial x^2}$$

where λ represent the diffusion coefficient. In the short hand notations:

$$\lambda u_{xx} - u_t = 0 \qquad \text{---------- (15)}$$

Note that similar to the wave equation, for the diffusion equation also, *x* and *t* are independent variables. Now, comparing equation (15) and (9) provides the coefficients for the diffusion equation:

A= λ, B=0 and C=0

Then the descriminant is:

$$\Delta = B^2 - 4\,AC = 0$$

As a result, for the diffusion equation the descriminant $\Delta = 0$. This means that *the diffusion equation is always parabolic.*

Some other examples:

(1) $$\frac{\partial^2 u}{\partial x^2}+y\frac{\partial^2 u}{\partial y^2}+\frac{\partial u}{\partial x}+\frac{\partial u}{\partial y}=0$$

Short notation:

$$u_{xx}+y\,u_{yy}+u_x+u_y=0 \qquad \text{----------} \ (16)$$

Compare equations (16) with equation (9) and we can get:

$$A=1,\ B=0 \text{ and } C=y$$

Then the descriminant is:

$$\Delta = B^2 - 4AC = 0-4y = -4y$$

Equation (16) is hyperbolic when y <0, parabolic when y= 0, and elliptic when y >0.

(2) $$x\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial x \partial y}+\frac{\partial^2 u}{\partial y^2}+\frac{\partial u}{\partial x}+\frac{\partial u}{\partial y}=0$$

or $$x u_{xx}+u_{xy}+u_{yy}+u_x+u_y=0 \qquad \text{----------} \ (17)$$

A comparison of equation (17) and equation (9) provides us:

$$A= x,\ B=1 \text{ and } C=1$$

Then the descriminant is:

$$\Delta = B^2 - 4AC = 1-4x$$

Equation (17) is hyperbolic when x < 1/4, parabolic when x= 1/4, and elliptic when x >1/4.

Thanks for the attention!