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# Energy spectrum of a free Dirac particle - pair production and pair annihilation.

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The Dirac relativistic equation for a free particle is: -

$$i\,\hbar\frac{\partial\psi}{\partial t} = c\,\hbar\overrightarrow{\alpha}.\,\vec{p}\,\psi + \beta mc^2\psi \quad \ldots\ldots (1)$$

With 4x 4 matrices for $\alpha$ and $\beta$ and a column vector with four rows for ψ, Dirac's equation is equivalent to four simultaneous first order differential equations that are linear and homogeneous in the four components $\psi_1, \psi_2, \psi_3$ and $\psi_4$.

The plane wave solution is

$$\psi_j = u_j e^{i(\vec{k}.\vec{r}-\omega t)} \qquad j= 1,2,3,4 \qquad \ldots\ldots. (2)$$

$\psi_j$'s are eigenfunctions of energy and momentum with eigenvalues E = ħ$\omega$ and p= kħ respectively.

The Dirac wavefunction ψ must be a four-column vector:

$$\Psi = \begin{pmatrix}\psi_1\\ \psi_2\\ \psi_3\\ \psi_4\end{pmatrix} \qquad \ldots\ldots (3)$$

Putting equation (2) and (3) in equation (1), we get

$$E\,\Psi = c\,(\overrightarrow{\alpha}.\,\vec{p})\,\psi + \beta mc^2\psi$$

$$Eu.\,e^{i(\vec{k}.\vec{r}-\omega t)} = c\,(\overrightarrow{\alpha}.\,\vec{p})\,u e^{i(\vec{k}.\vec{r}-\omega t)} + \beta mc^2 u e^{i(\vec{k}.\vec{r}-\omega t)}$$

$$Eu = c\,(\overrightarrow{\alpha}.\,\vec{p})u + \beta mc^2 u \qquad \ldots\ldots. (4)$$

Where u is a four-column vector partitioned into two components as:

$$u = \begin{pmatrix}u_1\\ u_2\\ u_3\\ u_4\end{pmatrix} = \begin{pmatrix}v\\ w\end{pmatrix}, \text{ where } v = \begin{pmatrix}u_1\\ u_2\end{pmatrix}, \quad w = \begin{pmatrix}u_3\\ u_4\end{pmatrix} \quad \ldots..(5)$$

putting equation (5) in equation (4), we get

$$E\begin{pmatrix} v \\ w \end{pmatrix} = c\,(\vec{\alpha}.\ \vec{p})\begin{pmatrix} v \\ w \end{pmatrix} + \beta mc^2 \begin{pmatrix} v \\ w \end{pmatrix} \quad \ldots\ldots (6)$$

Where $\alpha_i = \begin{pmatrix} 0 & \delta_i \\ \delta_i & 0 \end{pmatrix}, \beta = \begin{pmatrix} I & 0 \\ 0 & -I \end{pmatrix}$

And $\vec{\alpha} = \begin{pmatrix} 0 & \vec{\delta} \\ \vec{\delta} & 0 \end{pmatrix}$

$\vec{\delta}$ *is* $2x2$ *matrix* and $v$ *and* $w$ are two component column vectors.

Then equation (6) becomes

$$E\begin{pmatrix} v \\ w \end{pmatrix} = c\begin{pmatrix} 0 & \vec{\delta}.\vec{p} \\ \vec{\delta}.\vec{p} & 0 \end{pmatrix}\begin{pmatrix} v \\ w \end{pmatrix} + mc^2 \begin{pmatrix} I & 0 \\ 0 & -I \end{pmatrix}\begin{pmatrix} v \\ w \end{pmatrix}$$

$$\begin{pmatrix} Ev \\ Ew \end{pmatrix} = c\begin{pmatrix} (\vec{\delta}.\vec{p}) & w \\ (\vec{\delta}.\vec{p}) & v \end{pmatrix} + mc^2 \begin{pmatrix} v \\ -w \end{pmatrix}$$

$$Ev = c(\vec{\delta}.\vec{p})w + mc^2 v \quad \ldots\ldots (7)$$

$$Ew = c(\vec{\delta}.\vec{p})v - \ mc^2 w \quad \ldots\ldots (8)$$

Rearranging equation (7) and (8), we get

$$(E - mc^2)\ v = c(\vec{\delta}.\vec{p})w \quad \ldots. (9)$$

And, $(E + mc^2)\ w = c(\vec{\delta}.\vec{p})v \quad \ldots\ldots (10)$

Multiplying equation (9) by $(E + mc^2)$

$$(E + mc^2)(E - mc^2)v = c(\vec{\delta}.\vec{p})(E + mc^2)w$$

$$(E^2 - m^2c^4)v = c^2(\vec{\delta}.\vec{p})^2 v \quad \ldots.. (11)$$

To solve further we have to know $(\vec{\delta}.\vec{p})$

$$\vec{\delta}.\vec{p} = \delta_x p_x + \delta_y p_y + \delta_z p_z$$

$$\vec{\delta}.\vec{p} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} p_x & 0 \\ 0 & p_x \end{pmatrix} + \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}\begin{pmatrix} p_y & 0 \\ 0 & p_y \end{pmatrix} + \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\begin{pmatrix} p_z & 0 \\ 0 & p_z \end{pmatrix}$$

$$= \begin{pmatrix} 0 & p_x \\ p_x & 0 \end{pmatrix} + \begin{pmatrix} 0 & -ip_y \\ ip_y & 0 \end{pmatrix} + \begin{pmatrix} p_z & 0 \\ 0 & -p_z \end{pmatrix}$$

$$\vec{\delta}.\vec{p} = \begin{pmatrix} p_z & p_x - ip_y \\ p_x + ip_y & -p_z \end{pmatrix}$$

And $$\left(\vec{\delta}.\vec{p}\right)^2 = \begin{pmatrix} p_z & p_x - ip_y \\ p_x + ip_y & -p_z \end{pmatrix}\begin{pmatrix} p_z & p_x - ip_y \\ p_x + ip_y & -p_z \end{pmatrix}$$

$$= \begin{pmatrix} p_z^2 + p_x^2 + p_y^2 & 0 \\ 0 & p_z^2 + p_x^2 + p_y^2 \end{pmatrix}$$

$$\left(\vec{\delta}.\vec{p}\right)^2 = \begin{pmatrix} p^2 & 0 \\ 0 & p^2 \end{pmatrix} = p^2 \qquad .... (12)$$

Putting this equation in (11), we get

$$(E^2 - m^2c^4)v = c^2p^2v$$

$$\Rightarrow \qquad (E^2 - m^2c^4 - c^2p^2)v = 0$$

$v \neq 0$ otherwise solution will be zero, for non- trivial solution

$$E^2 = m^2c^4 + c^2p^2$$

$$E^2 = c^2p^2 + m^2c^4$$

$$E = \pm\sqrt{c^2p^2 + m^2c^4} \quad \text{energy for free particle}$$

Dirac equation gives positive as well as negative in both positive and negative energy solutions are possible.

Now we determine the energy eigenfunctions. The independent solutions are two for positive and two for negative energy.

From equation (9)

$$v = \frac{c\left(\vec{\delta}.\vec{p}\right)w}{(E - mc^2)}$$

When p= 0, $E_+ = mc^2$

We cannot use equation (9) for positive energy because $v = \infty$. So we use equation (10) with $E_+$. equation (10) becomes

$$w = \frac{c\left(\vec{\delta}.\vec{p}\right)v}{(E_+ + mc^2)}$$

For convenience, we can consider $v$ as

$$v = \begin{pmatrix}1\\0\end{pmatrix} \text{ and } \begin{pmatrix}0\\1\end{pmatrix}$$

We get two independent solutions for $E_+$. From equation (10), first we take with $v = \begin{pmatrix}1\\0\end{pmatrix}$

$$w = \frac{c\left(\vec{\delta}.\vec{p}\right)}{(E_+ + mc^2)}\begin{pmatrix}1\\0\end{pmatrix} \qquad \text{...... (13)}$$

And using relation

$$\vec{\delta}.\vec{p} = \begin{pmatrix} p_z & p_x - ip_y \\ p_x + ip_y & -p_z \end{pmatrix}$$

Putting this relation in equation (13),

$$w = \frac{c}{(E_+ + mc^2)}\begin{pmatrix} p_z & p_x - ip_y \\ p_x + ip_y & -p_z \end{pmatrix}\begin{pmatrix}1\\0\end{pmatrix}$$

$$= \frac{c}{(E_+ + mc^2)}\begin{pmatrix} p_z \\ p_x + ip_y \end{pmatrix}$$

$$w = \begin{pmatrix} \frac{cp_z}{(E_+ + mc^2)} \\ \frac{c(p_x + ip_y)}{(E_+ + mc^2)} \end{pmatrix} \quad \text{when } v = \begin{pmatrix}1\\0\end{pmatrix} \qquad \text{........(14)}$$

And From equation (10), Secondly we take with $v = \begin{pmatrix}0\\1\end{pmatrix}$

$$w = \frac{c\left(\vec{\delta}.\vec{p}\right)}{(E_+ + mc^2)}\begin{pmatrix}0\\1\end{pmatrix}$$

$$= \frac{c}{(E_+ + mc^2)} \begin{pmatrix} p_z & p_x - ip_y \\ p_x + ip_y & -p_z \end{pmatrix} \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

$$w = \frac{c}{(E_+ + mc^2)} \begin{pmatrix} p_x - ip_y \\ -p_z \end{pmatrix}$$

$$w = \begin{pmatrix} \frac{c(p_x - ip_y)}{(E_+ + mc^2)} \\ \frac{-cp_z}{(E_+ + mc^2)} \end{pmatrix} \quad \text{.......} \ (15)$$

Thus, for the eigenvalue E₊, the two independent solutions are $u^{(1)} = \begin{pmatrix} v \\ w \end{pmatrix} =$

$$\begin{pmatrix} 1 \\ 0 \\ \frac{cp_z}{(E_+ + mc^2)} \\ \frac{c(p_x + ip_y)}{(E_+ + mc^2)} \end{pmatrix} , \ u^{(2)} = \begin{pmatrix} v \\ w \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \\ \frac{c(p_x - ip_y)}{(E_+ + mc^2)} \\ \frac{-cp_z}{(E_+ + mc^2)} \end{pmatrix}$$

....... (16)

Now corresponding to $E_-$, With equation (9) the eigenfunctions for

$$E_- = \sqrt{c^2p^2 + m^2c^4}$$

So, equation (9) becomes,

$$v = \frac{c\left(\vec{\delta}.\vec{p}\right)w}{(E_- - mc^2)}$$

We will suppose $w = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ and $\begin{pmatrix} 0 \\ 1 \end{pmatrix}$,in the same way for $E_-$, *then we get the two independent solution is*

$$\text{Firstly, } w = \begin{pmatrix} 1 \\ 0 \end{pmatrix} , u^{(3)} = \begin{pmatrix} \frac{cp_z}{(E_- - mc^2)} \\ \frac{c(p_x + ip_y)}{(E_- - mc^2)} \\ 1 \\ 0 \end{pmatrix} \quad \text{.......} \ (17)$$

Secondly, $w$ = $\begin{pmatrix} 0 \\ 1 \end{pmatrix}$ , $u^{(4)} = \begin{pmatrix} \frac{c(p_x - ip_y)}{(E_- - mc^2)} \\ \frac{-cp_z}{(E_- - mc^2)} \\ 0 \\ 1 \end{pmatrix}$ ....... (18)

Therefore, $E_+ \cong mc^2$ and $E_- \cong -mc^2$.

Equation (9) and (10) consists of two equation. Hence $E_+$and $E_-$ occur twice. when p=0, $E_+ = mc^2$ and $E_- = -mc^2$.The energy spectrum of a free particle has two branches corresponding to$E_+$and $E_-$i.e one starting at $mc^2$ and extending to +∞ $as\ |p| \to \infty$ and the other starting at - $mc^2$ and extending to -∞ $as\ |p| \to \infty$.

The two branches are separated by a forbidden gap of width 2 $mc^2$. $No\ energy$ levels exists in the forbidden group.

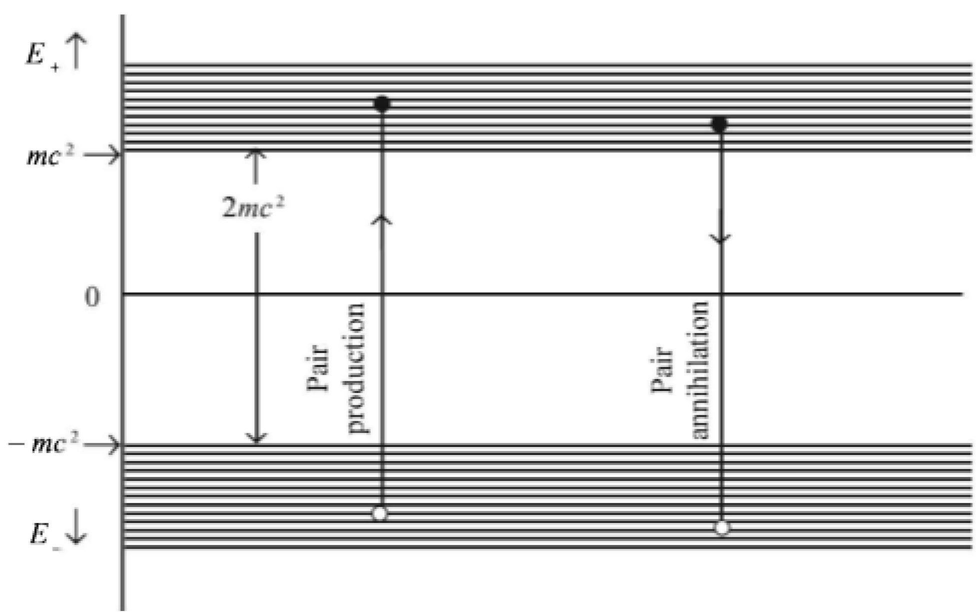


Fig. shows that energy levels of a free Dirac particle

If negative energy solutions represented accessible negative energy particle states, that all positive energy electrons would fall spontaneously into these lower energy states. Clearly this does not occur.

Dirac proposed that the vacuum corresponds to the state where all negative energy states are occupied. In the Dirac sea picture, the

Pauli exclusion principle prevents positive energy electrons from falling into the fully occupied negative energy states. Furthermore, a photon with energy E >$2mc^2$ could excite an electron from a negative energy state, leaving a hole in the vacuum. A hole in the vacuum would correspond to a state with more energy (less negative energy) and a positive charge relative to the fully occupied vacuum. In this way, holes in the Dirac sea correspond to positive energy antiparticles with the opposite charge to the particle states. The Dirac sea interpretation thus provides a picture for $e^+e^-$ pair production and also particle–antiparticle annihilation.

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