Quantum Field Theory

M.Sc. 4th Semester

MPHYEC-1: Advanced Quantum Mechanics

Unit III (Part 5)

Topic: System of Bosons

Dr. Ashok Kumar Jha

Assistant Professor

Department of Physics, Patna University

Mob:7903067108, Email: ashok.jha1984@gmail.com

The N- representation
(System of Bosons)

One could easily expand Ψ in terms of some complete orthonormal set of functions $\{u_k\}$ which carry all the space dependence of Ψ leaving the operator properties of Ψ to be expressed through the expansion coefficients which depend on the time:

$$\Psi(r,t) = \sum a_k(t) u_k(r) \quad \ldots (1)$$

Eq. (5) of Part 4: Quantization of Schrodinger Equation, now takes the form:

$$\Psi^\dagger(r,t) = \frac{1}{i\hbar}\pi(r,t) = \sum a_k^\dagger(t) u_k^*(\mathrm{r}) \quad \ldots (2)$$

The most convenient choice for $u_k$is the set of single particle energy eigenfunctions which satisfy,

$$-\frac{\hbar^2}{2m}\nabla^2 u_k + V u_k = E_k u_k \quad \ldots (3)$$

The coefficients $a_k(t)$ and $a_k^\dagger(\mathrm{t})$ are operators and suitable commutation relations for them have to be obtained.

SYESTEM OF BOSONS:

Multiplying eqn. (1) by $u_l^*(\mathrm{r})$ and integrating over the whole range of the variable

$$\int u_l^*(\mathrm{r})\Psi(\mathrm{r,t})\mathrm{d}^3 r = \sum a_k(t) \int u_l{}^*(r) u_k(r) d^3 r$$

Using orthonormality of the u_ks

$$a_k(t) = \int u_k^*(r)\Psi(r,t) d^3 r \quad \ldots (4)$$

Similarly, one can show that

$$a_k^\dagger(t) = \int u_k(r)\Psi^\dagger(r,t) d^3 r \quad \ldots (5)$$

The commutator a_k with $a_k^\dagger$ is:

$$[a_k, a_l^\dagger] = \iint u_k^*(r) u_l(r') d^3 r d^3 r' \delta(r - r') \quad \ldots (6)$$

$$= \delta_{kl} \quad \ldots (7)$$

In a similar way,

$$[a_k, a_l] = [a_k^\dagger, a_l^\dagger] = 0 \quad \ldots (8)$$

It is obvious from commutation relations that the amplitudes a_k and $a_k^\dagger$,infinite in numbers, are behaving as operators.

Another useful operator called number operator, representing the total number of particles is defined by,

$$N = \int \Psi^{\dagger} \Psi d^3 r \qquad \dots (9)$$

Substituting eqn. (13) and (14) gives,

$$N = \sum_k \sum_l a_k{}^{\dagger} a_l \int u_k^* (r) u_l(r) d^3 r$$

$$= \sum_k \sum_l a_k{}^{\dagger} a_l \delta_{kl}$$

$$= \sum_k N_k \qquad \dots (10)$$

Where $N_k = a_k a_k^{\dagger}$ … (11)

We shall now show the each N_k commutates with all others

$$[N_k, N_l] = [a_k^{\dagger} a_k, a_l^{\dagger} a_l]$$

$$= [a_k^{\dagger} a_k, a_l^{\dagger}] a_l + a_l^{\dagger} [a_k^{\dagger} a_k, a_l]$$

$$= 0 \qquad \dots (12)$$

Since each N_k commutates with all others, they can have simultaneous eigenkets and can be diagonalized simultaneously. Labelling the eigenkets by the eigenvalues $n_1, n_2 \dots \dots, n_k, \dots \dots., \infty$, the states of the quantized field in the representation in which each N_k is diagonal are the kets,

$$|n_1, n_2 \dots \dots, n_k, \dots\rangle$$

Next let us find the eigen values of the operator N_k. Its eigen value eqn. is:

$$N_k \boldsymbol{\Psi}(n_k) = n_k \boldsymbol{\Psi}(n_k) \qquad \dots (13)$$

Where n_k is the eigenvalue. Multiplying the eqn from left by $\Psi^{\dagger}(n_k)$ and integrating over the entire space.

$$n_k = \int \Psi^{\dagger}(n_k) \; N_k \boldsymbol{\Psi}(n_k) d^3 r = \int \Psi^{\dagger}(n_k) a_k^{\dagger} a_k \boldsymbol{\Psi}(n_k) d^3 r$$

$$= \int |a_k \boldsymbol{\Psi}(n_k)|^2 d^3 r \geq 0 \qquad \dots (14)$$

That is, the eigen values of N_k are all positive integers including zero:

$$n_k = 0, 1, 2, \dots, \infty \qquad \dots. (15)$$

Since the lower eigenvalue of N_k is zero, there must exist an eigenket $|0\rangle$ such that $N_k|0\rangle = 0$ for all k. the lowest normalized eigenket with no particle in state $|0\rangle$ is called vaccum state.

To understand the significance of the operator N_k substitute the value of $\Psi(r,t)$, eqn. (13) in the field Hamiltonian H, eqn. (7):

$$H = \sum_k \sum_l a_k{}^{\dagger} a_l \int \left(\frac{\hbar^2}{2m} . \nabla u_k^* \nabla u_l + V u_k^* u_l \right) d^3 r \qquad \dots (16)$$

Integrating the first term by parts we have,

$$\int \nabla u_k^* \nabla u_l d^3r = \int u_k^* \nabla u_l ds - \int u_k^* \nabla^2 u_l \, d^3r \quad \text{.... (17)}$$

Since $u_k \to 0$ at infinite bounding surface, the first term on the right side vanishes, Consequently,

$$H = \sum_k \sum_l a_k^{\dagger} a_l \int u_k^* \left(-\frac{\hbar^2}{2m} . \nabla^2 u_l + V u_l\right) d^3r \quad \text{... (18)}$$

Using Schr0dinger Equation,

$$-\frac{\hbar^2}{2m}\nabla^2 u_l + V u_l = E_l u_l$$

$$H = \sum_k \sum_l a_k^{\dagger} a_l \int u_k^* E_l u_l d^3r$$

$$= \sum_k \sum_l a_k^{\dagger} a_l E_l \int u_k^* u_l d^3r$$

$$= \sum_k \sum_l a_k^{\dagger} a_l E_k = \sum_k N_k E_k \quad \text{... (19)}$$

The eigen value of Hamiltonian H of the field in the state $|n_1, n_2 \dots \dots, n_k, \dots\rangle$ is,

$$E = \langle H \rangle = \sum_k n_k E_k \quad \text{... (20)}$$

It is evident from eqn. (32) that n_k is the number of particles in the state u_k with energy E_k and hence N_k can be regarded as the particle number operator in the k^{th} state. This justifies the name number operator forN. Since a given state $u_k$can be occupied by any number of particles of the same energy, the field represents an assembly of bosons.

Reference:

1. An Introduction to Quantum Field Theory by Mrinal Dasgupta
2. QUANTUM FIELD THEORY *A Modern Introduction* by Michio Kaku
3. First Book of Quantum Field Theory by Amitabha Lahiri & P. B. Pal
4. Quantum mechanics by G.S. Chaddha