Radiation from an Oscillating Electric Dipole

Course: MPHYEC-01 Plasma Physics
(M.Sc. Sem-IV)

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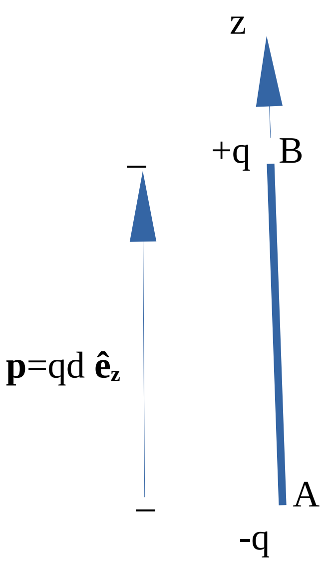
Unit-03

<u>Oscillating Electric Dipole</u>

To understand the meaning of oscillating electric dipole, first we take two opposite charges located at a distance *d* along z-direction at points A and B. Basically, this pairs of charges form an electric dipole with associated dipole moment (**p)** being directed along z-direction (as shown in figure 1).



Now, if we somehow change the charges at point A and B with time. Moreover, if the change is such that charges at point A and B become the oscillatory function of time i.e., for example:

$$q(t)= q_0 \sin(\omega t)$$

where ω represents the angular frequency of the oscillation. In such a situation, the dipole moment associated with the dipole is:

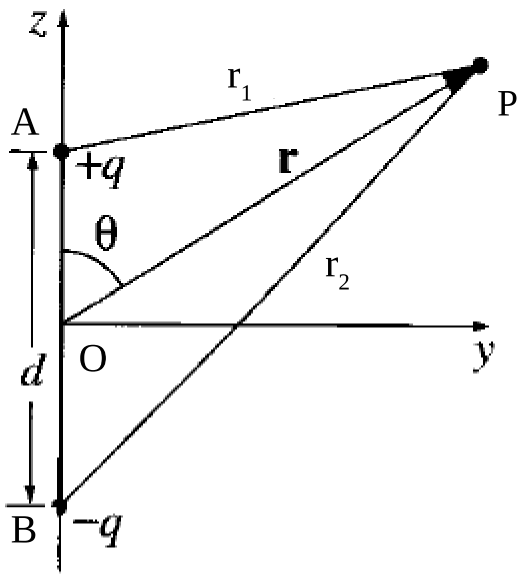
$$\mathbf{p}(t)=q_0 d \sin(\omega t)\,\hat{\mathbf{e}}_z$$

$$\mathbf{p}(t)=p_0 \sin(\omega t)\,\hat{\mathbf{e}}_z$$

where $p_0 = q_0 d$ represents the maximum value of dipole moment. From the above equation, it is clear that the dipole moment also becomes the oscillatory function of time. *Such an electric dipole with the associated dipole moment oscillates with time is known as an oscillatory electric dipole.*

Radiation from an Oscillating Electric Dipole

To calculate the radiation emitted by an oscillating electric dipole, we take an oscillating dipole system as shown in the below figure. Now, our aim is determine the expression for the radiation at point P located at r_1 distance from +q charge and r_2 distance from -q charge (see the figure). Moreover, the position vector **r** represents the position of point P with respect to the center O of the dipole (midpoint on the axis of the dipole). The vector **r** is oriented at angle θ with the axis.



Now, from the previous lectures, the retarded potential due to +q charge at point P is:

$$V_1 = \frac{1}{4\pi\varepsilon_0}\frac{q_0\cos[\omega(t-\frac{r_1}{c})]}{r_1} \qquad \text{------ (1)}$$

In the above expression, potential is calculated at retared time $t_r = t - r_1/c$.

Similarly, the retarded potential due to -q charge at point P is

$$V_2 = -\frac{1}{4\pi\varepsilon_0}\frac{q_0\cos[\omega(t-\frac{r_2}{c})]}{r_2} \qquad \text{------ (2)}$$

The net retarded potential due to both the charges is:

$$V(\boldsymbol{r},t)=\frac{1}{4\pi\varepsilon_0}\left(\frac{q_0\cos[\omega(t-r_1/c)]}{r_1}-\frac{q_0\cos[\omega(t-r_2/c)]}{r_2}\right) \qquad \text{----- (3)}$$

Now note that for the triangle AOP, length of the sides are AO=d/2, OP=r, AP=r_1. So from the law of cosines (also called as the cosine rule) for the triangle AOP:

$$r_1^{\,2}=r^2+(d/2)^2-2r(d/2)\cos\theta$$

$$r_1=\sqrt{r^2+d^2/4-r\,d\cos\theta} \qquad \text{----- (4)}$$

Similarly, for the triangle BOP, using the law of cosines, we can have:

$$r_2^{\,2}=r^2+(d/2)^2-2r(d/2)\cos(\pi-\theta)$$

$$r_2=\sqrt{r^2+d^2/4+r\,d\cos\theta} \qquad \text{----- (5)}$$

For the dipole to be perfect, we assume r>>d (i.e. distance between charges is much smaller than the distance of observation point from the center of the dipole). Under this approximation, from equation (4), we can write:

$$r_1=r\sqrt{1+\frac{d^2}{4r^2}-\frac{d\cos\theta}{r}}$$

as (d/r) <<1, we can use the binomial expansion $(1+a)^{1/2}$ = 1+(1/2)a+... and can only consider first term and neglect higher order term. Then

$$r_1\approx r\left(1+\frac{d^2}{8r^2}-\frac{d}{2r}\cos\theta\right)$$

Since (d/r) <<1, so we can neglect $d^2/8r^2$ term in the above equation and we get:

$$r_1\approx r\left(1-\frac{d}{2r}\cos\theta\right) \qquad \text{----- (6)}$$

Using the similar approximation, from equation (5), we can write:

$$r_2\approx r\left(1+\frac{d}{2r}\cos\theta\right) \qquad \text{----- (7)}$$

Now, using equation (6)

$$\cos[\omega(t-r_1/c)]\approx\cos[\omega(t-\frac{r}{c}(1-\frac{d}{2r}\cos\theta))]$$

$$\cos[\omega(t-r_1/c)]\approx\cos[\omega(t-\frac{r}{c})+\frac{\omega d}{2c}\cos\theta]$$

$$\cos[\omega(t-r_1/c)]\approx\cos[\omega(t-\frac{r}{c})]\cos(\frac{\omega d}{2c}\cos\theta)-\sin[\omega(t-\frac{r}{c})]\sin(\frac{\omega d}{2c}\cos\theta) \quad \text{--- (8)}$$

Similarly, from equation (7), we can get

$$\cos[\omega(t-r_2/c)]\approx\cos[\omega(t-\frac{r}{c})]\cos(\frac{\omega d}{2c}\cos\theta)+\sin[\omega(t-\frac{r}{c})]\sin(\frac{\omega d}{2c}\cos\theta) \quad \text{--- (9)}$$

Next, **the another approximation associated with a perfect oscillating electric dipole is d<<(c/ω) or (ωd/c)<<1.** Since, the wavelength associated with a wave having angular frequency ω, λ = 2π/k=2πc/ω; then this approximation corresponds to d << λ. Under this approximation, (ωd/c)<<1,

$$\cos(\frac{\omega d}{2c}\cos\theta)\approx 1$$

and $\sin(\frac{\omega d}{2c}\cos\theta)\approx\frac{\omega d}{2c}\cos\theta$ (as for small θ, cosθ=1 and sinθ=θ)

Putting these two relations in equations (8) and (9), we obtain

$$\cos[\omega(t-r_1/c)]\approx\cos[\omega(t-\frac{r}{c})]-\frac{\omega d}{2c}\cos\theta\sin[\omega(t-\frac{r}{c})] \quad \text{---- (10)}$$

$$\cos[\omega(t-r_2/c)]\approx\cos[\omega(t-\frac{r}{c})]+\frac{\omega d}{2c}\cos\theta\sin[\omega(t-\frac{r}{c})] \quad \text{---- (11)}$$

Now, from equation (6), we can write:

$$\frac{1}{r_1}\approx\frac{1}{r}(1-\frac{d}{2r}\cos\theta)^{-1}$$

Once again using the binomial expansion $(1-a)^{-1}$ = 1+a-... and only retaining the first term, we get

$$\frac{1}{r_1}\approx\frac{1}{r}(1+\frac{d}{2r}\cos\theta) \quad \text{----- (12)}$$

In the similar way, using equation (7), we can have

$$\frac{1}{r_2} \approx \frac{1}{r}\left(1-\frac{d}{2r}\cos\theta\right) \qquad \text{----- (13)}$$

Utilizing equations (10), (11), (12) and (13) in equation (3), we get an expression of scalar potential of a perfect oscillating electric dipole:

$$V(\boldsymbol{r},t)=\frac{q_0}{4\pi\varepsilon_0}\left(\frac{1}{r}\left(1+\frac{d}{2r}\cos\theta\right)\left(\cos\left[\omega\left(t-\frac{r}{c}\right)\right]-\frac{\omega d}{2c}\cos\theta\sin\left[\omega\left(t-\frac{r}{c}\right)\right]\right)\right)$$
$$-\frac{q_0}{4\pi\varepsilon_0}\left(\frac{1}{r}\left(1-\frac{d}{2r}\cos\theta\right)\left(\cos\left[\omega\left(t-\frac{r}{c}\right)\right]+\frac{\omega d}{2c}\cos\theta\sin\left[\omega\left(t-\frac{r}{c}\right)\right]\right)\right)$$

$$V(\boldsymbol{r},t)=\frac{q_0}{4\pi\varepsilon_0 r}\left(\frac{d}{r}\cos\theta\cos\left[\omega\left(t-\frac{r}{c}\right)\right]-\frac{\omega d}{c}\cos\theta\sin\left[\omega\left(t-\frac{r}{c}\right)\right]\right)$$

$$V(\boldsymbol{r},t)=\frac{q_0 d\cos\theta}{4\pi\varepsilon_0 r}\left(\frac{1}{r}\cos\left[\omega\left(t-\frac{r}{c}\right)\right]-\frac{\omega}{c}\sin\left[\omega\left(t-\frac{r}{c}\right)\right]\right)$$

$$V(\boldsymbol{r},t)=\frac{p_0\cos\theta}{4\pi\varepsilon_0 r}\left(\frac{1}{r}\cos\left[\omega\left(t-\frac{r}{c}\right)\right]-\frac{\omega}{c}\sin\left[\omega\left(t-\frac{r}{c}\right)\right]\right) \qquad \text{----- (14)}$$

<u>Remark:</u> *Note that in the limit ω --> 0 means $q=q_0$, this corresponds to the stationary dipole. For such a dipole, equation (14) modifies as:*

$$V(\boldsymbol{r},t)=\frac{p_0\cos\theta}{4\pi\varepsilon_0 r}\left(\frac{1}{r}\times 1-\frac{\omega}{c}\times 0\right)$$

$$V(\boldsymbol{r},t)=\frac{p_0\cos\theta}{4\pi\varepsilon_0 r^2}$$

This is well-known expression for the potential for the dipole. But, the stationary dipole doesn't radiate.

To simplify the calculations, **an additional approximation is made: r >> (c/ω). Since λ = 2π/k=2πc/ω, this approximation corresponds to r>> λ.** Under this approximation, |(1/r) cos[ω(t-r/c)]| << |(ω/c) sin[ω(t-r/c)]|. As a result, the first term in equation (14) can be neglected and equation (14) modifies as:

$$V(\boldsymbol{r},t) = -\frac{p_0 \omega \cos\theta}{4\pi\varepsilon_0 rc} \sin[\omega(t - \frac{r}{c})] \qquad \text{------ (15)}$$

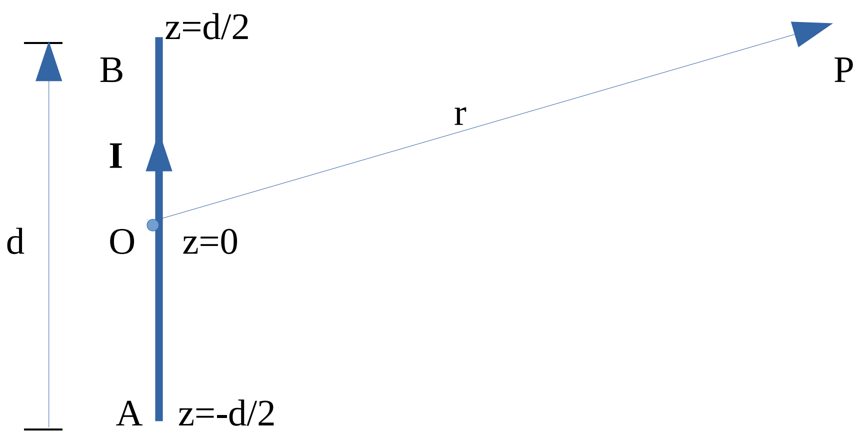
In order to obtain an expression of electric field and magnetic field, our next aim is to calculate vector potential. For this, first we note that the current flowing along the axis of the dipole due to the oscillatory charges is:

$$\boldsymbol{I}(t) = \frac{dq}{dt} \hat{\boldsymbol{e}}_{\boldsymbol{z}} \qquad \text{(since } q = q_0 \cos(\omega t))$$

$$\boldsymbol{I}(t) = -q_0 \omega \sin(\omega t) \hat{\boldsymbol{e}}_{\boldsymbol{z}} \qquad \text{------ (16)}$$

As we know that vector potential associated with a wire carrying current (**I**) of length L is

$$\boldsymbol{A}(\boldsymbol{r},t) = \frac{\mu_0}{4\pi} \int_{-L/2}^{L/2} \frac{\boldsymbol{I}\, dl}{r} \qquad \text{------ (17)}$$



We can treat the oscillating dipole as a current carrying wire of length d oriented along z-axis such that the center O of the dipole is located at the origin z=0. Then the coordinate of one end A of wire is z=-d/2 and the coordinate of another end B is z=-d/2 (see above figure). And we can consider length element dl = dz. The current passing through the dipole is given by equation (16). Using all these inputs in equation (17) to calculate the retarded vector potential associate with the dipole,

$$\boldsymbol{A}(\boldsymbol{r},t) = \frac{\mu_0}{4\pi} \int_{-d/2}^{d/2} \frac{-q_0 \omega \sin[\omega(t - r/c)] \hat{\boldsymbol{e}}_{\boldsymbol{z}} dz}{r}$$

Note that as we are calculate retarded vector potential, that's why time t is replaced by t-(r/c) in the above expression.

$$A(r,t) = -\frac{\mu_0 q_0 \omega \sin[\omega(t-r/c)]\hat{e}_z}{4\pi r} \int_{-d/2}^{d/2} dz \quad \text{(using above approximations)}$$

$$A(r,t) = -\frac{\mu_0 p_0 \omega}{4\pi r} \sin[\omega(t-r/c)]\hat{e}_z \qquad \text{------ (18)}$$

After calculating potentials V(**r**, t) and **A**(**r**, t), we can deduce the expressions for electric field and magnetic field using following relations:

$$E(r,t) = -\nabla V - \frac{\partial A}{\partial t} \qquad \text{------ (19)}$$

$$B(r,t) = \nabla \times A \qquad \text{------ (20)}$$

For evaluating the required gradient and curl of the potentials, for this problem, it is convient to use the spherical coordinate system. From equation (15), V(**r**, t)=V(r, θ, t) which means the scalar potential only depends on position coordinates r and θ and doesn't depend on φ, therefore, we have

$$\nabla V = \frac{\partial V}{\partial r}\hat{r} + \frac{1}{r}\frac{\partial V}{\partial \theta}\hat{\theta}$$

$$= -\frac{p_0 \omega}{4\pi \epsilon_0 c}\left\{\cos\theta\left(-\frac{1}{r^2}\sin[\omega(t-r/c)] - \frac{\omega}{rc}\cos[\omega(t-r/c)]\right)\hat{\mathbf{r}} - \frac{\sin\theta}{r^2}\sin[\omega(t-r/c)]\hat{\boldsymbol{\theta}}\right\}$$

Under the approximation r>> (c/ω), in the above equation, first and the last terms can be neglected and we get,

$$\nabla V \approx \frac{p_0 \omega^2}{4\pi \varepsilon_0 c^2}\left(\frac{\cos\theta}{r}\right)\cos[\omega(t-r/c)]\hat{r}$$

$$\nabla V \approx \frac{p_0 \mu_0 \omega^2}{4\pi}\left(\frac{\cos\theta}{r}\right)\cos[\omega(t-r/c)]\hat{r} \qquad \text{------ (21)}$$

Taking partial time derivative of equation (18), we get

$$\frac{\partial A(r,t)}{\partial t} = -\frac{\mu_0 p_0 \omega^2}{4\pi r}\cos[\omega(t-r/c)]\hat{e}_z$$

Using coordinate transformation $\hat{\mathbf{e}}_z = \cos\theta\ \hat{\mathbf{r}} - \sin\theta\ \hat{\boldsymbol{\theta}}$, from equation (18), vector potential can be re-written as:

$$\boldsymbol{A}(\boldsymbol{r},t) = -\frac{\mu_0 p_0 \omega}{4\pi r}\sin[\omega(t-r/c)](\cos\theta\,\hat{\boldsymbol{r}} - \sin\theta\,\hat{\boldsymbol{\theta}}) \qquad \text{------ (22)}$$

Then

$$\frac{\partial \boldsymbol{A}(\boldsymbol{r},t)}{\partial t} = -\frac{\mu_0 p_0 \omega^2}{4\pi r}\cos[\omega(t-r/c)](\cos\theta\,\hat{\boldsymbol{r}} - \sin\theta\,\hat{\boldsymbol{\theta}}) \qquad \text{------ (23)}$$

Using equations (21) and (23) into equation (19), we get the electric field

$$\boldsymbol{E}(\boldsymbol{r},t) = -\frac{\mu_0 p_0 \omega^2}{4\pi}\cos[\omega(t-r/c)]\left(\frac{\sin\theta}{r}\right)\hat{\boldsymbol{\theta}} \qquad \text{------ (24)}$$

For magnetic field, we need to calculate

$$\begin{aligned}\nabla \times \mathbf{A} &= \frac{1}{r\sin\theta}\left(\frac{\partial}{\partial\theta}(A_\varphi \sin\theta) - \frac{\partial A_\theta}{\partial\varphi}\right)\hat{\mathbf{r}} \\ &+ \frac{1}{r}\left(\frac{1}{\sin\theta}\frac{\partial A_r}{\partial\varphi} - \frac{\partial}{\partial r}(rA_\varphi)\right)\hat{\boldsymbol{\theta}} \qquad \text{------ (25)} \\ &+ \frac{1}{r}\left(\frac{\partial}{\partial r}(rA_\theta) - \frac{\partial A_r}{\partial\theta}\right)\hat{\boldsymbol{\varphi}},\end{aligned}$$

From equation (22), components of the vector potential are:

$$A_r(r,\theta) = -\frac{\mu_0 p_0 \omega}{4\pi r}\sin[\omega(t-r/c)]\cos\theta$$

$$A_\theta(r,\theta) = \frac{\mu_0 p_0 \omega}{4\pi r}\sin[\omega(t-r/c)]\sin\theta$$

$$A_\varphi = 0$$

By straightforward inspection of the expressions of the above components and equation (25), it is clear that only φ-component of curl of **A** is non-zero. Hence,

$$\nabla\times\boldsymbol{A} = \frac{1}{r}\left(\frac{\partial}{\partial r}(r A_\theta) - \frac{\partial A_r}{\partial\theta}\right)\hat{\boldsymbol{\varphi}}$$

Use the compoents of **A** in the above expression, we get

$$\nabla\times\boldsymbol{A}=-\frac{\mu_0 p_0 \omega}{4\pi r}\left(\frac{\omega}{c}\sin\theta\cos[\omega(t-r/c)]+\frac{\sin\theta}{r}\sin[\omega(t-r/c)]\right)\hat{\boldsymbol{\varphi}} \qquad \text{------ (26)}$$

Under the approximation r>> (c/ω), 2nd term in the above equation can be neglected, then magnetic field

$$\boldsymbol{B}(\boldsymbol{r},t)=\nabla\times\boldsymbol{A}=-\frac{\mu_0 p_0 \omega}{4\pi r}\left(\frac{\omega}{c}\sin\theta\cos[\omega(t-r/c)]\right)\hat{\boldsymbol{\varphi}}$$

$$\boldsymbol{B}(\boldsymbol{r},t)=-\frac{\mu_0 p_0 \omega^2}{4\pi c}\cos[\omega(t-r/c)]\left(\frac{\sin\theta}{r}\right)\hat{\boldsymbol{\varphi}} \qquad \text{------ (27)}$$

<u>Remark:</u> *From equations (24) and (27), it is clear that **E** and **B** are oscillatory functions and both are in phase. In addition, **E** and **B** are perpendicular to each other and |**E**|/|**B**|=c. It means that **E** and **B** represent (spherical) electromagnetic waves in free space.*

Now, to calculate the energy radiated by the oscillating electric dipole, we deduce the Poynting vector:

$$\boldsymbol{S}=\frac{1}{\mu_0}(\boldsymbol{E}\times\boldsymbol{B})$$

Use **E** and **B** from equations (24) and (27) for calculating **S**

$$\boldsymbol{S}=\frac{\mu_0}{c}\left\{\frac{p_0\omega^2}{4\pi}\cos[\omega(t-r/c)]\left(\frac{\sin\theta}{r}\right)\right\}^2\hat{\boldsymbol{r}}$$

Then, the Poynting vector averaged for a full time period is,

$$\langle\boldsymbol{S}\rangle=\left(\frac{\mu_0 p_0^2\omega^4}{32\pi^2 c}\right)\frac{\sin^2\theta}{r^2}\hat{\boldsymbol{r}} \qquad \text{------- (28)}$$

where the time averaged value of $\langle\cos^2[\omega(t-r/c)]\rangle=1/2$ is used.

The power radiated by the dipole is then given by:

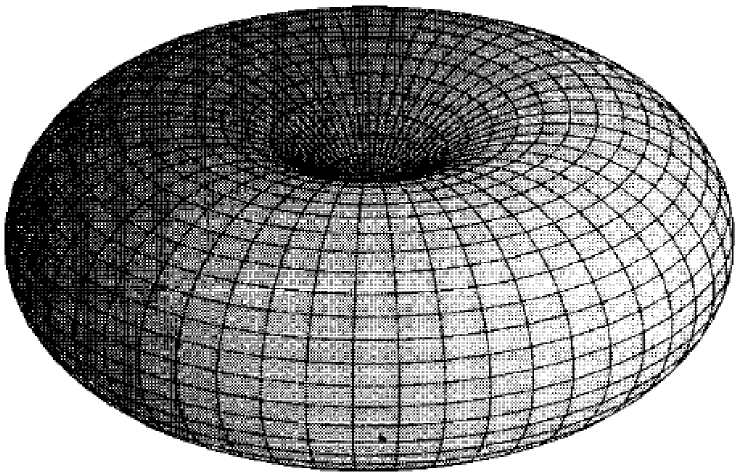
$$\langle P\rangle=\int\langle\boldsymbol{S}\rangle.d\boldsymbol{a} \qquad \text{------- (29)}$$

Use equation (28) into equation (29), we get

$$\langle P \rangle = \left(\frac{\mu_0 p_0^2 \omega^4}{32\pi^2 c}\right) \int \frac{\sin^2\theta}{r^2} r^2 \sin\theta \, d\theta \, d\varphi$$

$$\langle P \rangle = \frac{\mu_0 p_0^2 \omega^4}{12\pi c} \qquad \text{------- (30)}$$

From equation (28), it is clear that power radiated along the axis of the dipole is zero as sinθ=0 because θ=0 along the axis. The radiation by the dipole is predominantly emitted in the space which is of the donut shape as shown in figure below.



Reference: “Introduction to Electrodynamics” by David J. Griffiths

Thanks for the attention!