Quantum Field Theory
M.Sc. 4th Semester
MPHYEC-1: Advanced Quantum Mechanics
Unit III (Part 3)
Topic: Quantisation of Field

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Quantisation of Field:

To Quantise the field, the field variables ψ and π are treated as operation functions. Just as quantum conditions,

$$[q_i, q_k]=[p_i, p_k]=0 \text{ and } [q_i,p_k]= i\hbar\delta_{ik} \quad \ldots (1)$$

Were used for the transition from classical to quantum particle mechanics, we achieve the transition from classical to quantum field theory by requiring that

$$[\psi_i, \psi_j]=[P_i, P_j]=0 \text{ and } [\psi_i,P_j]= i\hbar\delta_{ij} \quad \ldots (2)$$

Assuming the cell volumes are very small equation (2) can be written in terms of ψ and π in the following forms:

$$[\psi(\vec{r},t), \psi(\vec{r}',t)]= [\pi(\vec{r},t), \pi(\vec{r}',t)] = 0 \quad \ldots(3a)$$

$$[\psi(\vec{r},t), \pi(\vec{r}',t)]= i\hbar\delta(\vec{r},\vec{r}') \quad \ldots(3b)$$

where $\delta(\vec{r},\vec{r}') = 1/\delta\tau_i$ if $\vec{r}$ and $\vec{r}'$ are in the same cell and zero otherwise. In the limit, the cellvolume approach zero,

$\delta(\vec{r},\vec{r}')$ can be replaced by the three-dimensional Dirac δ function $\delta(\vec{r}-\vec{r}')$. The quantum condition for the canonical field variables ψ and π then becomes,

$$[\psi(\vec{r},t), \psi(\vec{r}',t)]= [\pi(\vec{r},t), \pi(\vec{r}',t)] = 0 \quad \ldots(4a)$$

$$[\psi(\vec{r},t), \pi(\vec{r}',t)]= i\hbar\delta(\vec{r}-\vec{r}') \quad \ldots (4b)$$

By making ψ and π non-commuting operators, we convert H, L etc., also into operators; which have eigenvalues, eigenstates etc.

Again, using the earlier concept of quantisation described during Lagrangian formulation of field, the equation of motion for any quantum dynamical variable F is obtained by replacing the Poisson bracket by the commutator bracket divided by $i\hbar$ then,

$$\frac{d\hat{F}}{dt} = \frac{[\hat{F},\hat{H}]}{i\hbar} + \frac{\partial\hat{F}}{\partial t} \quad \ldots (5)$$

Quantizing the Complex Scaler Field:

To analyse the QFT of a (free) complex scalar the Lagrangian of this system is

$$\mathcal{L} = (\partial_\mu\varphi^*)(\partial^\mu\varphi) - m^2|\varphi|^2. \quad \ldots (1)$$

The fields φ and φ^* are complex-conjugates. We work in the mostly minus convention for the Makowski metric,

$$\eta = (1, -1, -1, -1)$$

The conjugate momenta are,

$$\pi = \frac{\partial \mathcal{L}}{\delta \partial_0 \phi} = \partial_0 \phi^*, \Pi^* = \frac{\delta \mathcal{L}}{\delta \partial O \phi^*} = \partial_0 \phi \quad \ldots (2)$$

Thus, the Hamiltonian density is,

$$\begin{aligned}\mathcal{H} =& \pi^* \dot{\phi}^* + \pi \dot{\phi} - \mathcal{L}(\phi, \partial_0 \phi) = |\pi|^2 - \left((\partial_\mu \phi^*)(\partial^\mu \phi) - m^2 |\phi|^2\right) \\ =& \pi^*(\vec{x}, t)\pi(\vec{x}, t) + \vec{\nabla}\phi^*(\vec{x}, t) \cdot \vec{\nabla}\phi(\vec{x}, t) + m^2 |\phi(\vec{x}, t)|^2\end{aligned} \quad \ldots (3)$$

and the Hamiltonian is therefore,

$$H = \int \mathcal{H} d^3x = \int d^3x \;\; \pi^*(x,t)\pi(x,t) + \nabla\, \varphi^*(x,t) \cdot \nabla\varphi(x,t) + m^2 |\varphi(x,t)|^2 \quad \ldots (4)$$

Let us define,

$$\phi = \frac{1}{\sqrt{2}}(\phi_1 + i\phi_2) \quad \ldots (5)$$

$$\Pi = \frac{1}{\sqrt{2}}\left(\dot{\phi}_1 - i\phi_2\right) \quad \ldots (6)$$

with φ^*, π^* their complex conjugate, so that

$$\begin{aligned}\phi_1 =& \frac{1}{\sqrt{2}}(\phi + \phi^*) \\ \pi_1 =& \frac{1}{\sqrt{2}}(\pi + \pi^*) \\ \phi_2 =& -\frac{i}{\sqrt{2}}(\phi - \phi^*) \\ \pi_2 =& -\frac{i}{\sqrt{2}}(\pi^* - \pi)\end{aligned} \quad \ldots (7)$$

Performing canonical quantization means imposing the canonical commutations relations between φ, φ^* with their conjugate momenta. This leads to,

$$[\phi_1(\vec{x},t), \pi_1(\vec{y},t)] = \frac{1}{2}[\phi(\vec{x},t) + \phi(\vec{x},t)^*, \pi(\vec{y},t) + \pi^*(\vec{y},t)] = \frac{1}{2}(i\delta(\vec{x}-\vec{y}) + i\delta(\vec{x}-\vec{y})) = i\delta(\vec{x}-\vec{y}) \quad \ldots (8)$$

and more generally

$$[\varphi_I(x, t), \pi_i(y, t)] = i\delta_{ij}\delta(x - y) \quad \ldots (9)$$

The Hamiltonian now decouples

$$H = \frac{1}{2}\int d^3x \; (\pi_1 + i\pi_2)(\pi_1 - i\pi_2) + \vec{\nabla}(\phi_1 - i\phi_2) \cdot \vec{\nabla}(\phi_1 + i\phi_2) + m^2|\phi_1 + i\phi_2|^2 \quad \ldots (10)$$

$$=\frac{1}{2}\int d^3x \sum_{i=1}^{2}\pi_i^2+\left(\vec{\nabla}(\phi_i)\right)^2+m^2|\phi_i|^2=H_1+H_2 \qquad \dots (11)$$

We can quantize each of these as seen in class. The equations of motion are

$$\dot{\phi_1}(\vec{x},t)=-i[\phi_1(\vec{x},t),H_1]=-i\int d^3y\left[\phi_1(\vec{x},t),\frac{1}{2}\pi(\vec{y},t)^2+\frac{1}{2}(\vec{\nabla}\phi_1(\vec{y},t))^2+\frac{1}{2}m^2\phi_1^2(\vec{y},t)\right]$$
$$=-2i\frac{1}{2}\int d^3y\pi_1(\vec{y},t)i\delta^{(3)})(\vec{x}-\vec{y})=\pi_1(\vec{x},t) \qquad \dots (12)$$

$$\dot{\pi_1}\vec{x},t=-i[\pi_1(\vec{x},t),H_1]=-i\int d^3y\left[\pi_1(\vec{x},t),\partial_i\phi_1(\vec{y},t)\right]\partial_i\phi_1(\vec{y},t)\left[\pi(\vec{x},t),\phi(\vec{y},t)\right]\phi(\vec{y},t)$$
$$=-i\int d^3y\partial_i\phi_1(\vec{y},t)\partial_i\left(-i\delta^{(3)}(\vec{x}-\vec{y})\right)+m^2\phi(\vec{y},t)\left(-i\delta^{(3)}(\vec{x}-\vec{y})\right)$$
$$=-\int d^3y\left(-\partial_i\partial^i\phi(\vec{x},t)+m^2\phi(\vec{x},t)\right)\delta^{(3)}(\vec{x}-\vec{y})=\nabla^2\phi_1(\vec{x},t)-m^2\phi_1(\vec{x},t). \qquad \dots (13)$$

The last equality is obtained using integration by parts. So, overall, we have

$$\partial_t^2\,\varphi_1=\partial_t^2\pi_1=\nabla^2\,\varphi_1+m^2\,\varphi_1 \qquad \dots(14)$$

which is just the Klein-Gordon equation

$$(\partial_t^{\,2}-\nabla^2+m^2)\,\varphi_1=0. \qquad \dots(15)$$

Taking the Fourier Transform of this equation,

$$\phi_1=\int\frac{d^3p}{(2\pi)^3}e^{i\vec{p}\vec{x}}\tilde{\phi}_1(\vec{p},\quad) \qquad \dots (16)$$

leads to the equation,

$$(\partial_t^{\,2}+p^2+m^2)\,\tilde{\varphi}_1=0. \qquad \dots(17)$$

The modes with different momenta are decoupled in momentum space, and we have the equation of motion of a Harmonic Oscillator (HO) with $\omega_p=\sqrt{p^2+m^2}$ at every p. That is, the Klein-Gordon equation is equivalent to an infinite number of Harmonic Oscillators. The solution, as you well know, is

$$\tilde{\phi}_1(\vec{p},t)=\left(a_p^{(1)}e^{-i\omega_p t)}+a_p^{(2)}e^{i\omega_p t)}\right)\sqrt{\frac{1}{2\omega_p}} \qquad \dots (18)$$

The field φ_1 is real, and this means that $\tilde{\phi}_1(\vec{p},t)=\tilde{\phi}_1^*(-\vec{p},t)$ Therefore, we obtain

$$\tilde{\phi}_1(\vec{p},t)=\left(a_pe^{-i\omega_p t)}+a_{-p}^*e^{i\omega_p t)}\right)\sqrt{\frac{1}{2\omega_p}} \qquad \dots (19)$$

Upon quantization, we promote the coefficients a_p to operators (replacing complex conjugation with Hermitian conjugation) obtaining

$$\phi_1(\vec{x},t) = \int \frac{d^3p}{(2\pi)^3}\frac{1}{\sqrt{2\omega_p}}\left(a_p e^{i(\vec{p}\vec{x}-\omega_p t)} + a_p^\dagger e^{-i(\vec{p}\vec{x}-\omega_p t)}\right)$$

$$\pi_1(\vec{x},t) = \dot{\phi}_1 = -i\int \frac{d^3p}{(2\pi)^3}\sqrt{\frac{\omega_p}{2}}\left(a_p e^{i(\vec{p}\vec{x}-\omega_p t)} - a_p^\dagger e^{-i(\vec{p}\vec{x}-\omega_p t)}\right) \quad \ldots (20)$$

If we now invert the Fourier transform and use the canonical commutation relations to

$$[\varphi_1(x, t), \pi_1(y, t)] = i\delta(x - y),$$

we will see that ,

$$[a_p, a_{p'}^\dagger] = (2\pi)^3\delta(p - p^0) \quad \ldots(21)$$

Equivalently, we will show that this commutation relation leads to the canonical one

$$[\phi_1(\vec{x},t), \pi_1(\vec{y},t)] = -\frac{i}{2}\int \frac{d^3p}{(2\pi)^3}\frac{d^3p'}{(2\pi)^3}[a_p, a_{p'}^\dagger]e^{i(\vec{p}\vec{x}-\vec{p}'\vec{y})-i(\omega_p-\omega_{p'})t} + [a_p^\dagger, a_{p'}]e^{-i(\vec{p}\vec{x}-\vec{p}'\vec{y})+i(\omega_p-\omega_{p'})t}$$

$$= -\frac{i}{2}\int \frac{d^3p}{(2\pi)^3}\frac{d^3p'}{(2\pi)^3}(2\pi)^3\delta(p-p')\left(-e^{i(\vec{p}\vec{x}-\vec{p}'\vec{y})-i(\omega_p-\omega_{p'})t} - e^{-i(\vec{p}\vec{x}-\vec{p}'\vec{y})+i(\omega_p-\omega_{p'})t}\right)$$

$$= i\int \frac{d^3p}{(2\pi)^3}e^{ip(x-y)} = i\delta(x \quad) \quad \ldots (22)$$

We can repeat this procedure for the real field φ_2, and obtain

$$\phi_1(\vec{x},t) = \int \frac{d^3p}{(2\pi)^3}\frac{1}{\sqrt{2\omega_p}}\left(b_p e^{i(\vec{p}\vec{x}-\omega_p t)} + b_p^\dagger e^{-i(\vec{p}\vec{x}-\omega_p t)}\right)$$

$$\pi_1(\vec{x},t) = \dot{\phi}_1 = -i\int \frac{d^3p}{(2\pi)^3}\sqrt{\frac{\omega_p}{2}}\left(b_p e^{i(\vec{p}\vec{x}-\omega_p t)} - b_p^\dagger e^{-i(\vec{p}\vec{x}-\omega_p t)}\right) \quad \ldots (23)$$

Upon returning to the original complex field we have

$$\phi(\vec{x},t) = \frac{1}{\sqrt{2}}(\phi_1 + i\phi_2) = \int \frac{d^3p}{(2\pi)^3}\frac{1}{\sqrt{2\omega_p}}\left(\underbrace{\left(\frac{a_p + ib_p}{\sqrt{2}}\right)}_{A_p} e^{i(\vec{p}\vec{x}-\omega_p t)} + \underbrace{\left(\frac{a_p^\dagger + ib_p^\dagger}{\sqrt{2}}\right)}_{B_p^\dagger} e^{-i(\vec{p}\vec{x}-\omega_p t)}\right) \quad \ldots (24)$$

$$\phi^\dagger(\vec{x},t) = \frac{1}{\sqrt{2}}(\phi_1 - i\phi_2) = \int \frac{d^3p}{(2\pi)^3}\frac{1}{\sqrt{2\omega_p}}\left(\underbrace{\left(\frac{a_p - ib_p}{\sqrt{2}}\right)}_{B_p} e^{i(\vec{p}\vec{x}-\omega_p t)} + \underbrace{\left(\frac{a_p^\dagger - ib_p^\dagger}{\sqrt{2}}\right)}_{A_p^\dagger} e^{-i(\vec{p}\vec{x}-\omega_p t)}\right) \quad \ldots (25)$$

It is easy to verify that these new fields satisfy,

$$[A_p, B_{p'}] = [A_p^\dagger, B_{p'}^\dagger] = [A_p, B_{p'}] = [B_p, B_{p'}] = 0 \qquad \text{… (26)}$$

$$[A_p, A_{p'}^\dagger] = [B_p, B_{p'}^\dagger] = (2\pi)^3 \delta^{(3)}(\vec{p} - \vec{p'}). \qquad \text{… (27)}$$

Computing the Hamiltonian:

We have, with these definitions, the following mode expansion

$$\phi(\vec{x}, t) = \int \frac{d^3p}{(2\pi)^3} \frac{1}{\sqrt{2\omega_p}} \left\{ A_p e^{i(\vec{p}\vec{x} - \omega_p t)} + B_p^\dagger e^{-i(\vec{p}\vec{x} - \omega_p t)} \right\} \qquad \text{…(28)}$$

$$\phi^\dagger(\vec{x}, t) = \int \frac{d^3p}{(2\pi)^3} \frac{1}{\sqrt{2\omega_p}} \left\{ B_p e^{i(\vec{p}\vec{x} - \omega_p t)} + A_p^\dagger e^{-i(\vec{p}\vec{x} - \omega_p t)} \right\} \qquad \text{…(29)}$$

$$\pi(\vec{x}, t) = -i \int \frac{d^3p}{(2\pi)^3} \sqrt{\frac{\omega_p}{2}} \left\{ B_p e^{i(\vec{p}\vec{x} - \omega_p t)} - A_p^\dagger e^{-i(\vec{p}\vec{x} - \omega_p t)} \right\} \qquad \text{…(30)}$$

$$\pi^\dagger(\vec{x}, t) = -i \int \frac{d^3p}{(2\pi)^3} \sqrt{\frac{\omega_p}{2}} \left\{ A_p e^{i(\vec{p}\vec{x} - \omega_p t)} - B_p^\dagger e^{-i(\vec{p}\vec{x} - \omega_p t)} \right\} \qquad \text{…(31)}$$

with $\omega_p = \sqrt{p^2 + m^2}$. The Hamiltonian of a real scalar field is

$$H = \int \frac{d^3p}{(2\pi)^3} \omega_p \left[a_p^\dagger a_p \right] + const\ . \qquad \text{… (32)}$$

Therefore, the Hamiltonian of the complex field is (ignoring the constant)

$$\begin{aligned} H &= \int \frac{d^3p}{(2\pi)^3} \omega_p \left[a_p^\dagger a_p + b_p^\dagger b_p \right] \\ &= \int \frac{d^3p}{(2\pi)^3} \omega_p \left[A_p^\dagger A_p + B_p^\dagger B_p) \right] \end{aligned} \qquad \text{… (33)}$$

Now, some comments are in order

- The operator φ destroys A quanta and creates B's. $\varphi^\dagger$ does the opposite.
- The vacuum is defined by $A| 0 > = B | 0 > = 0$. • There is a two-fold degeneracy of the spectrum

Notice that the theory, (specifically, the Lagrangian density), is invariant under $\varphi \leftarrow e^{i\alpha} \varphi$. In terms of φ_1 and φ_2 this is

$$\begin{pmatrix} \phi_1 \\ \phi_2 \end{pmatrix} = O \begin{pmatrix} \phi_1 \\ \phi_2 \end{pmatrix} \qquad \text{… (34)}$$

where O_{ij} is a 2×2 orthogonal matrices.

The conserved charge for the U (1) symmetry is

$$Q = i\int d^3x \left(\phi\partial_t\phi^\dagger - \phi^\dagger\partial_t\phi\right) = i\int d^3x\left(\phi\pi - \phi^\dagger\pi^\dagger\right) \qquad \ldots (35)$$

In terms of the creation and annihilation operators, we get

$$Q = i\int \frac{d^3p d^3k}{(2\pi)^6}\frac{i\sqrt{\omega_p}}{2\sqrt{\omega_k}}\int d^3x\left[\left(A_k e^{i(\vec{k}\vec{x}-\omega_k t)} + B_k^\dagger e^{-i(\vec{p}\vec{k}-\omega_k t)}\right)\left(A_p^\dagger e^{-i(\vec{p}\vec{x}-\omega_p t)} - B_p e^{i(\vec{p}\vec{x}-\omega_p t)}\right)\right.$$
$$\left. - \left(A_k^\dagger e^{-i(\vec{k}\vec{x}-\omega_k t)} + B_k e^{i(\vec{k}\vec{x}-\omega_k t)}\right)\left(-A_p e^{i(\vec{p}\vec{x}-\omega_p t)} + B_p^\dagger e^{-i(\vec{p}\vec{x}-\omega_p t)}\right)\right] \qquad \ldots (36)$$

Integrate over space. The mixed terms AB, $A^\dagger B^\dagger$ will cancel each other. For the other terms, use

$$\int\frac{d^3k}{(2\pi)^3}\int d^3x e^{i(\vec{k}\vec{x}-\omega_k t)-i(\vec{p}\vec{x}-\omega_p t)} = \int\frac{d^3k}{(2\pi)^3}e^{-i\omega_k t+i\omega_p t}\delta^{(3)}(\vec{p}-\vec{k}) = 1 \qquad \ldots (37)$$

We get,

$$Q = -\frac{1}{2}\int\frac{d^3p}{(2\pi)^3}\left(A_pA_p^\dagger - B_p^\dagger B_p + A_p^\dagger A_p - B_pB_p^\dagger\right) = \int\frac{d^3p}{(2\pi)^3}\left(A_p^\dagger A_p - B_p^\dagger B_p\right) \qquad \ldots (38)$$

The number of A excitations minus the number of B excitations is conserved. Since these are free fields, this is not a very interesting statement, but such symmetries also exist in interacting theories.

Some commutators

$$[H, A_p^\dagger] = \int\frac{d^3k}{(2\pi)^3}\omega_k A_k^\dagger[A_k, A_p^\dagger] = \omega_p A_p^\dagger$$
$$[H, A_p] = \int\frac{d^3k}{(2\pi)^3}\omega_k[A_k^\dagger, A_p]A_k = -\omega_p A_p \qquad \ldots (39)$$

and similarly, for B. The meaning is that $A^\dagger{}_p$ and $B_p{}^\dagger$ increase the energy by ω_p while A_p and B_p decrease the energy by ω_p. We can also compute the commutator with the charge

$[Q, A^\dagger{}_p] = A^\dagger{}_p$, $[Q, B_p{}^\dagger] = -B_p{}^\dagger$, $[Q, A_p] = -A_p$, $[Q, B_p] = B_p$. ...(40)

Therefore, the A-particle has charge 1 and the B-particle has charge -1.

Reference:

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2. QUANTUM FIELD THEORY *A Modern Introduction* by Michio Kaku
3. First Book of Quantum Field Theory by Amitabha Lahiri & P. B. Pal
4. Quantum mechanics by G.S. Chaddha