Electrostatic and Electromagnetic Waves in Plasma

Course: MPHYEC-01 Plasma Physics
(M.Sc. Sem-IV)

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Unit-03

Effective Plasma Permittivity

Before we study the wave in unmagnetized plasma, lets first define an important property of plasma called "effective plasma permittivity". For the purpose, we consider the following Maxwell's equation for the plasma

$$\nabla \times \boldsymbol{B} = \mu_0 \boldsymbol{J} + \mu_0 \varepsilon_0 \frac{\partial \boldsymbol{E}}{\partial t}$$

or

$$\nabla \times \boldsymbol{H} = \boldsymbol{J} + \varepsilon_0 \frac{\partial \boldsymbol{E}}{\partial t} \qquad \text{-------- (1)}$$

Now, in general, we can consider the form of electric field as:

$$\boldsymbol{E}(\boldsymbol{r}, t) = \boldsymbol{A}(\boldsymbol{r}) e^{(-i\omega t)} \qquad \text{-------- (2)}$$

With such a electric field, equation (1) can be re-written as:

$$\nabla \times \boldsymbol{H} = \boldsymbol{J} - i\omega\varepsilon_0 \boldsymbol{E}$$

and using the Ohm's law **J**=σ**E,** the above equation modifies as:

$$\nabla \times \boldsymbol{H} = \sigma \boldsymbol{E} - i\omega\varepsilon_0 \boldsymbol{E}$$

$$\nabla \times \boldsymbol{H} = -i\omega\varepsilon_0 \left(1 + \frac{i\sigma}{\omega\varepsilon_0}\right) \boldsymbol{E}$$

$$\nabla \times \boldsymbol{H} = -i\omega\varepsilon_0 \varepsilon_{eff} \boldsymbol{E} \qquad \text{-------- (3)}$$

where $\varepsilon_{eff} = 1 + \frac{i\sigma}{\omega\varepsilon_0}$ is called *effective plasma permittivity*.

To relate the property of plasma with a dielectric medium, we note that free electric current doesn't flow in the dielectric medium and, therefore, the Ampere's law for the dielectric medium is:

$$\nabla \times \boldsymbol{H} = \varepsilon \frac{\partial \boldsymbol{E}}{\partial t}$$

$$\nabla \times \boldsymbol{H} = -i\omega\varepsilon_0 \varepsilon_r \boldsymbol{E} \qquad \text{------- (4)}$$

Note that equations (3) and (4) are similar except ε_{eff} in equation (3) gets replaced with ε_r (dielectric constant/relative permittivity) in equation (4). This is the reason we called ε_{eff} as effective plasma permivittivity. Moreover, due to the similar structure of both the equations, we can also infer that plasma behaves like a dielectric medium with ε_{eff} being its dielectric constant.

The above discussed connection of plasma with dielectric medium can also be established by the 1st Maxwell's equation. We know that, in the absence of free charges, the first Maxwell's equation for the dielectric medium is given by:

$$\nabla . \boldsymbol{D} = 0$$

$$\nabla . (\varepsilon \boldsymbol{E}) = 0$$

$$\nabla . (\varepsilon_0 \varepsilon_r \boldsymbol{E}) = 0 \qquad \text{---------} \ (5)$$

Now the first Maxwell's equation for the plasma is:

$$\nabla . \boldsymbol{D} = \rho \qquad \text{---------} \ (6)$$

From the charge continuity equation:

$$\frac{\partial \rho}{\partial t} + \nabla . \boldsymbol{J} = 0$$

$$-i \omega \rho + \nabla . \sigma \boldsymbol{E} = 0$$

$$\rho = \frac{\nabla . \sigma \boldsymbol{E}}{i \omega} \qquad \text{---------} \ (7)$$

Using equation (7) in (6), we get

$$\nabla . (\varepsilon_0 \boldsymbol{E}) = \frac{\nabla . \sigma \boldsymbol{E}}{i \omega}$$

$$\nabla . \left(\varepsilon_0 \left(1 + \frac{i \sigma}{\omega \varepsilon_0}\right) \boldsymbol{E}\right) = 0$$

$$\nabla . (\varepsilon_0 \varepsilon_{eff} \boldsymbol{E}) = 0 \qquad \text{---------} \ (8)$$

Important is the similarity of equation (5) and (8). The similarity once again establishes that the behaviour of plasma is like a dielectric medium.

Electrostatic and Electromagnetic Waves in Plasma

To study the properties of electrostatic/electromagnetic waves in a plasma system, we write the following Maxwell's equations for the system:

$$\nabla\times\boldsymbol{E}=\frac{-\partial\,\boldsymbol{B}}{\partial t} \qquad \text{-------}\ (9)$$

$$\nabla\times\boldsymbol{H}=\boldsymbol{J}+\varepsilon_0\frac{\partial\,\boldsymbol{E}}{\partial t} \qquad \text{-------}\ (10)$$

Now to explore the wave propagation, we denote the variables (electric and magnetic field vectors) of the above equations in the plane wave form, i.e.

$$\boldsymbol{E}(\boldsymbol{r},t)=\boldsymbol{E_0}e^{i(\boldsymbol{k}.\boldsymbol{r}-\omega t)}$$

$$\boldsymbol{B}(\boldsymbol{r},t)=\boldsymbol{B_0}e^{i(\boldsymbol{k}.\boldsymbol{r}-\omega t)}$$

As a result, the differential equations (9) and (10) can be converted into algebric equations by replacing partial time derivative operator by -iω and position del operator by i**k**. After such a replacement equation (9) modifies as:

$$\boldsymbol{k}\times\boldsymbol{E}=\omega\mu_0\,\boldsymbol{H} \qquad \text{-------}\ (11)$$

and equation (10) with the use of equation (3) becomes:

$$\boldsymbol{k}\times\boldsymbol{H}=-\omega\,\varepsilon_0\,\varepsilon_{eff}\,\boldsymbol{E} \qquad \text{-------}\ (12)$$

From equation (11), we can write

$$\boldsymbol{k}\times\boldsymbol{k}\times\boldsymbol{E}=\omega\mu_0\,\boldsymbol{k}\times\boldsymbol{H} \qquad \text{-------}\ (13)$$

Using equation (12) into equation (13), we get

$$\boldsymbol{k}\times\boldsymbol{k}\times\boldsymbol{E}=-\omega^2\mu_0\,\varepsilon_0\,\varepsilon_{eff}\,\boldsymbol{E} \qquad \text{-------}\ (14)$$

$$\boldsymbol{k}(\boldsymbol{k}\,.\,\boldsymbol{E})-k^2\boldsymbol{E}=-\omega^2\mu_0\,\varepsilon_0\,\varepsilon_{eff}\,\boldsymbol{E} \qquad \text{-------}\ (15)$$

Equation (15) represents the wave equation in algebric form or the dispersion relation for the electromagnetic waves in the plasma.

Further, from equation (14) we can write

$$\boldsymbol{k}.(\boldsymbol{k}\times\boldsymbol{k}\times\boldsymbol{E})=-\omega^2\mu_0\varepsilon_0\varepsilon_{eff}\,\boldsymbol{k}.\boldsymbol{E}$$

Note that **k** is normal to (**k** x **k** x **E)** vector. Therefore, **k**.(**k** x **k** x **E)** =0 and above equation becomes:

$$\omega^2\mu_0\varepsilon_0\varepsilon_{eff}\,\boldsymbol{k}.\boldsymbol{E}=0$$

$$\varepsilon_{eff}\,\boldsymbol{k}.\boldsymbol{E}=0 \qquad \text{------ (16)}$$

From the above equation, there are two possible cases

Case (i) ε_{eff} =0 and **k.E** is non-zero.
Case (ii) **k.E**=0 and ε_{eff} is non-zero.

Lets first consider the case (i).

Case (i) ε_{eff} =0

With ε_{eff} =0 equation (14) becomes,

$$\boldsymbol{k}\times(\boldsymbol{k}\times\boldsymbol{E})=0$$

$$\boldsymbol{k}(\boldsymbol{k}.\boldsymbol{E})-k^2\boldsymbol{E}=0$$

$$\boldsymbol{k}(\boldsymbol{k}.\boldsymbol{E})=k^2\boldsymbol{E} \qquad \text{------ (17)}$$

*The above equation shows that **k** is parallel to **E** which means direction of wave propagation is along the electric field.*

As a result, **k** x **E =0** and equation (11) modifies as:

$$\omega\mu_0\boldsymbol{H}=0$$

$$\boldsymbol{H}=0$$

This shows that the magnetic field is zero.

*So, in the case of $\varepsilon_{eff}=0$, the wave will have oscillating electric field which is parallel to the wave progation (therefore the waves is longitudinal wave) and the wave doesn't support any magnetic field. Such waves are known as **electric** or **electrostatic waves**. Because of the oscillating electric field, there is also perturbation in charge particle density.*

Case (ii) k.E =0

For this case, from equation (15), we have

$$\boldsymbol{k}(\boldsymbol{k}.\boldsymbol{E}) - k^2 \boldsymbol{E} = -\omega^2 \mu_0 \varepsilon_0 \varepsilon_{eff} \boldsymbol{E}$$

$$k^2 \boldsymbol{E} = \omega^2 \mu_0 \varepsilon_0 \varepsilon_{eff} \boldsymbol{E}$$

$$k^2 = \frac{\omega^2}{c^2} \varepsilon_{eff} \qquad \text{------ (18)}$$

Now from equation (12), we have

$$\boldsymbol{k} \times \boldsymbol{H} = -\omega \varepsilon_0 \varepsilon_{eff} \boldsymbol{E}$$

This equation shows that **k** and **H** are normal to **E**. And from equation (11):

$$\boldsymbol{k} \times \boldsymbol{E} = \omega \mu_0 \boldsymbol{H}$$

k and **E** are normal to **H**. *Overall, for this case, wave propagation **k,** oscillating **E** and **H** are mutually perpendicular to each-other. The waves are transverse in nature. Therefore, the waves are **electromagnetic waves**.*

From equation (18), the phase speed is

$$\frac{\omega}{k} = \frac{c}{\sqrt{\varepsilon_{eff}}}$$

The speed is similar to the ones obtained in a dielectric medium. In the case of plasma, relative permittivity ε_r is replaced with ε_{eff}.

Thanks for the attention!