BORN APPROXIMATION

If the Scattering Centre is localised and the scattering is weak such that the scattered wave at large distance from the scatterer is weak in amplitude, then Born approximation can be applied to S-eqn. for scattering problem given by

$$(\nabla^2 + k'^2)\,\psi(r) = U(r)\,\psi(r) \quad — (1)$$

'OR'

$$\psi_K(r) = e^{iK\cdot r} + \frac{e^{iKr}}{4\pi r} f_K(\theta, \phi) \text{ With } — (2)$$

$$f_K(\theta,\phi) = -\frac{1}{4\pi}\int e^{-K'\cdot r'}\, U(r)\,\psi_K(r)\,dr \quad — (2a)$$

For solving these S-eqn. Born approximation can be used to obtain $f_K(\theta, \phi)$ and hence the differential cross-section.

$$\sigma(\theta,\phi) = |f(\theta,\phi)|^2$$

This Born approximation can be applied if V is fairly small and it uses method of successive approximations. Under this approximation rescattering of the scattered waves are neglected in comparision to incident ...

The difficulty in obtaining solution of S-eqn. for scattering problem is mainly due to appearance of $\psi_k(r)$ on the R.H.S. of eqn. (2) or (2a). To make it simple, Born approximation is applied and by process of iteration soln. of $\psi_k(r)$ in the form of ~~series~~ ~~solution~~ power series of $V(r)$ is realised.

By replacing r with r' in ~~above~~ wave function,

$$\psi_k(r') = e^{ik\cdot r'} - \frac{e^{ikr'}}{4\pi r'}\int e^{-ik''\cdot r''} U(r'')\,\psi_k(r'')\,dr'' \quad -(3)$$

and

$$\psi(r) = e^{ik\cdot r} - \frac{e^{ikr}}{4\pi r}\int e^{-ik'\cdot r'} U(r')\, e^{i(k\cdot r')}\,dr'$$

$$+ \frac{e^{ikr}}{4\pi r}\int e^{-ik'\cdot r'} U(r')\left\{\frac{e^{ikr'}}{4\pi r'}\int e^{-ik''\cdot r''} U(r'')\,\psi_k(r'')\,dr''\right\}dr' \quad -(4)$$

If $\psi_k(r'')$ is written as eqn. (3) and put in last term of eqn. (4), and the process of successive iteration is continued, we get a series of infinite terms.

Under Born approximation for weak interaction $V(r)$, the terms involving U^2 and other higher powers can be neglected.

$$\therefore \; \psi_k^{(1)}(r) = e^{ik\cdot r} - \frac{e^{ikr}}{4\pi r}\int e^{-ik'\cdot r'}\,U(r')\,e^{ik\cdot r'}\,dr' \qquad -(5)$$

and
$$\psi_k^{(2)}(r) = e^{ik\cdot r} - \frac{e^{ikr}}{4\pi r}\int e^{-ik'\cdot r'}\,U(r')\,e^{ik\cdot r'}\,dr' \qquad -(6)$$
$$+ \frac{e^{ikr}}{(4\pi)^2 r}\iint e^{-ik'\cdot r'}\,U(r')\,\frac{e^{ik''\cdot r'}}{r'}\,e^{-ik''r''}\,U(r'')\,e^{ik\cdot r''}\,dr'\,dr''$$

are called 1^{st} and 2^{nd} Born approximations respectively.

Eqn. (5) Can also be written as
$$\psi_k^{(1)}(r) = e^{ik\cdot r} + \frac{e^{ikr}}{r}\,f_k^{(1)}(\theta,\phi) \text{ with } f_k^{(1)}(\theta,\phi) = -\frac{1}{4\pi}\int e^{-ik'\cdot r}\,U(r)\,e^{ik\cdot r}\,dr = -\frac{1}{4\pi}\langle \phi_{k'}|U(r)|\phi_k\rangle$$

$f_k^{(1)}(\theta,\phi)$ is known as 1^{st} Born approximation to the scattering amplitude and is given by
$$f_k^{(1)}(\theta,\phi) = -\frac{m}{2\pi\hbar^2}\int V(r)\,e^{-i(k'-k)\cdot r}\,dr = -\frac{m\sqrt{2\pi}}{\hbar^2}\cdot V_q,$$
with V_q = q-th Fourier Component of Potential $V(r)$.

$$\Rightarrow \boxed{\sigma(\theta,\phi) = |f_k(\theta,\phi)|^2 = \left(\frac{m}{2\pi\hbar^2}\right)^2\left|\int e^{i(k'-k)\cdot r'}\,V(r')\,dr'\right|^2}$$

and $\hbar|k'-k|$ = Momentum transfer in Scattering.

Born approx. will be valid only if total wave function is not greatly different from incident wave $e^{ik\cdot r}$, i.e. when Scattered wave $\psi_s(r)$ is small Compared to incident in the region where $V(r)$ is large.

Mathematically, $|\psi_s(r)|^2 < 1$ for small value of r.

Method of Partial Waves and Phase Shift

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JANUARY

This method is mainly applicable to spherically symmetric potential, where scattering is completely symmetrical about the direction of incidence. The wave funct (and hence scattering amplitude) depends only on polar angle θ and not on azimuthal angle. In scattering problem V is a function of r and and $\psi_K(r)$ depends on K and r and angle between K and r only. Therefore, both $\psi_K(r,\theta)$ and $f_K(\theta)$ may be expanded in series of Legendre Polynomial $P_l(\cos\theta)$, which form a complete set. This method was originally applied by Rayleigh to the scattering of sound waves and then by Faxen and Holtsmark to the scattering of Schrödinger wave. A plane wave is equivalent to number of spherical waves and for spherically symmetric potential, the A.M. of scattered particle is a constant of motion. So, it is advantageous to express solutions in terms of A.M. eigenstates and with the expansion of $\psi_K(r,\theta)$ and $f(\theta)$ as

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$$\psi_K(r,\theta) = \sum_{l=0}^{\infty} R_l(K,r)\, P_l(\cos\theta) \quad \text{and}$$

$$f_K(\theta) = \sum_{l=0}^{\infty} f_l(K)\, P_l(\cos\theta)$$

The expansion Co-efficent $R_l(K,r)$ (a radial function) are Known as Partial Waves and $f_l(K)$ are Known as partial Waves amplitudes. The Subsitution of these expansions yield radial part of equation as.

$$\left[\frac{d^2}{dr^2} + \frac{2}{r}\frac{d}{dr} - \frac{l(l+1)}{r^2} - U(r) + K^2\right] R_l(K,r) = 0$$

or, $$\left[\frac{d^2}{dr^2} - \frac{l(l+1)}{r^2} - U(r) + K^2\right] U_l(K,r) = 0$$

with $U_l(K,r) = r R_l(K,r)$

For spherically Symmetric potential the radial wave fn. $R_l(K,r)$ is finite at $r=0$ and gives $U_l(K,0)=0$
For relatively Small ~~V(r)~~) Short ranged $V(r)$, we get Solution

$$R_l(K,r) \underset{r\to\infty}{\cong} A_l(K)\frac{1}{Kr}\sin\left(Kr - \frac{l\pi}{2} + \delta_l(K)\right)$$

with $A_l(K) = \left[B_l^2(K) + C_l^2(K)\right]^{1/2}$ and $\delta_l(K) = -\tan^{-1}\left[\frac{C_l(K)}{B_l(K)}\right]$

When $B_l(K)$ and $C_l(K)$ are independent of position r and are Co-efficients in Linear Combination Soln. of $R_l(K,r)$