Leap Frog Method Application-II: Motion of a Charged Particle in Magnetic field

Course: MPHYCC-05 Modeling and Simulation,
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Unit-3 of CC-05

Leap Frog Scheme

From the previous lecture, we know that for a typical 1D equation of motion in Mechanics:

$$a=\frac{dv}{dt}\,;\,v=\frac{dy}{dt} \quad \text{---------} \ (1)$$

the Leapfrog scheme is given as:

$$y_{i+1} = y_i + \Delta t \ v_{i+1/2} \quad \text{------------} \ (2)$$

$$v_{i+1/2} = v_{i-1/2} + \Delta t \ a_i \quad \text{------------}(3)$$

and to evaluate $v_{-1/2}$ from the given v_0, the used Euler's scheme is,

$$v_{-1/2} = v_0 - (\Delta t/2)\, a_0 \quad \text{-----------} \ (4)$$

Now as an another application of the scheme, in the following, we solve the equation of motion of a charged particle in a uniform magnetic field using this method.

Application of Leapfrog Method

Motion of a Charged Particle in a Uniform Magnetic field

Equation of motion of a charged particle q in a uniform 1D magnetic field **B=**$B_0\,\hat{e}_z$:

$$m\frac{d\boldsymbol{v}}{dt}=q(\boldsymbol{v}\times\boldsymbol{B}) \quad \text{----------}(5)$$

In components form:

$$\frac{dv_x}{dt}=\frac{qB_0}{m}v_y \quad ; \quad v_x=\frac{dx}{dt} \quad \text{---------} \ (6)$$

$$\frac{dv_y}{dt}=\frac{-qB_0}{m}v_x \quad ; \quad v_y=\frac{dy}{dt} \quad \text{---------} \ (7)$$

$$\frac{dv_z}{dt}=0 \quad ; \quad v_z=\frac{dz}{dt} \quad \text{---------} \ (8)$$

We need to solve the above equations of motion (equations (6)-(8)) with the specified initial velocity and position components $v_x(t=0)=v_{x0}$, $x(t=0)=x_0$, $v_y(t=0)=v_{y0}$, $y(t=0)=y_0$, $v_z(t=0)=v_{z0}$, $z(t=0)=z_0$. We can analytically solve the equation and show that the charged particle moves in a helical or circular path depending upon the angle between initial velocity **v** and magnetic field **B**. If we choose $v_{z0}= 0$, then the component of velocity parallel to the magnetic field is zero and **v** is normal to **B**, and, as a result, the charged particle traces circular trajectory. To simply the problem, we set $v_{z0}= 0$ and $z_0 =0$ and, consiquently, we don't need to solve equation (8). Therefore, we will be solving only equations (6) and (7).

Now, we want to solve equations (6)-(7) using the Leapfrog method. Applying Leapfrog scheme given by equations (2) & (3) for equations (6)-(7), we have

$$(v_x)_{i+1/2} = (v_x)_{i-1/2} + \Delta t \ (qB_0/m)(v_y)_{i-1/2} \quad \text{------ (9)}$$

$$x_{i+1} = x_i + \Delta t \ (v_x)_{i+1/2} \quad \text{------ (10)}$$

$$(v_y)_{i+1/2} = (v_y)_{i-1/2} - \Delta t \ (qB_0/m)(v_x)_{i-1/2} \quad \text{------ (11)}$$

$$y_{i+1} = y_i + \Delta t \ (v_y)_{i+1/2} \quad \text{------ (12)}$$

In addition, using equation (4), we do one time adjustment for obtaining velocities at $t=0-\Delta t/2$

$$(v_x)_{-1/2} = v_{x0} - (\Delta t/2) \ (qB_0/m) \ v_{y0} \quad \text{----------(13)}$$

$$(v_y)_{-1/2} = v_{x0} + (\Delta t/2) \ (qB_0/m) \ v_{x0} \quad \text{----------(14)}$$

Algorithm to Solve Eq^n of Motion of Charged Particle in Uniform Magnetic Field using Leapfrog Method

1. *Define the values of constant:* q, B_0, m

2. *Define initial time and time step:* t_0, Δt

3. *Define total number of steps:* n [*then final time* $t_f = t_0 + (n+1)\,\Delta t$]

4. *Specify initial conditions for* $t=t_0$: x_0, v_{x0}, y_0, v_{y0}, z_0, v_{z0}

5. *Calculate velocity* $v_{-1/2}$: $v_{x0} = v_{x0} - (\Delta t/2)\ (qB_0/m)\ v_{y0}$

$v_{y0} = v_{y0} + (\Delta t/2)\ (qB_0/m) v_{x0}$

6. *Start iteration (i=0, n)*

{ $v_{x1} = v_{x0} + (\Delta t)\ (qB_0/m)\ v_{y0}$

$v_{y1} = v_{x0} - (\Delta t)\ (qB_0/m)\ v_{x0}$

$x_1 = x_0 + \Delta t\ v_{x1}$

$y_1 = y_0 + \Delta t\ v_{y1}$

write x_1, y_1, v_{x1}, v_{y1}

set $x_0 = x_1$, $y_0 = y_1$, $v_{x0} = v_{x1}$, $v_{y0} = v_{y1}$ }

7. *end*

C-Program to Solve Eq^{n} of Motion of Charged Particle in Uniform Magnetic Field using Leapfrog Method

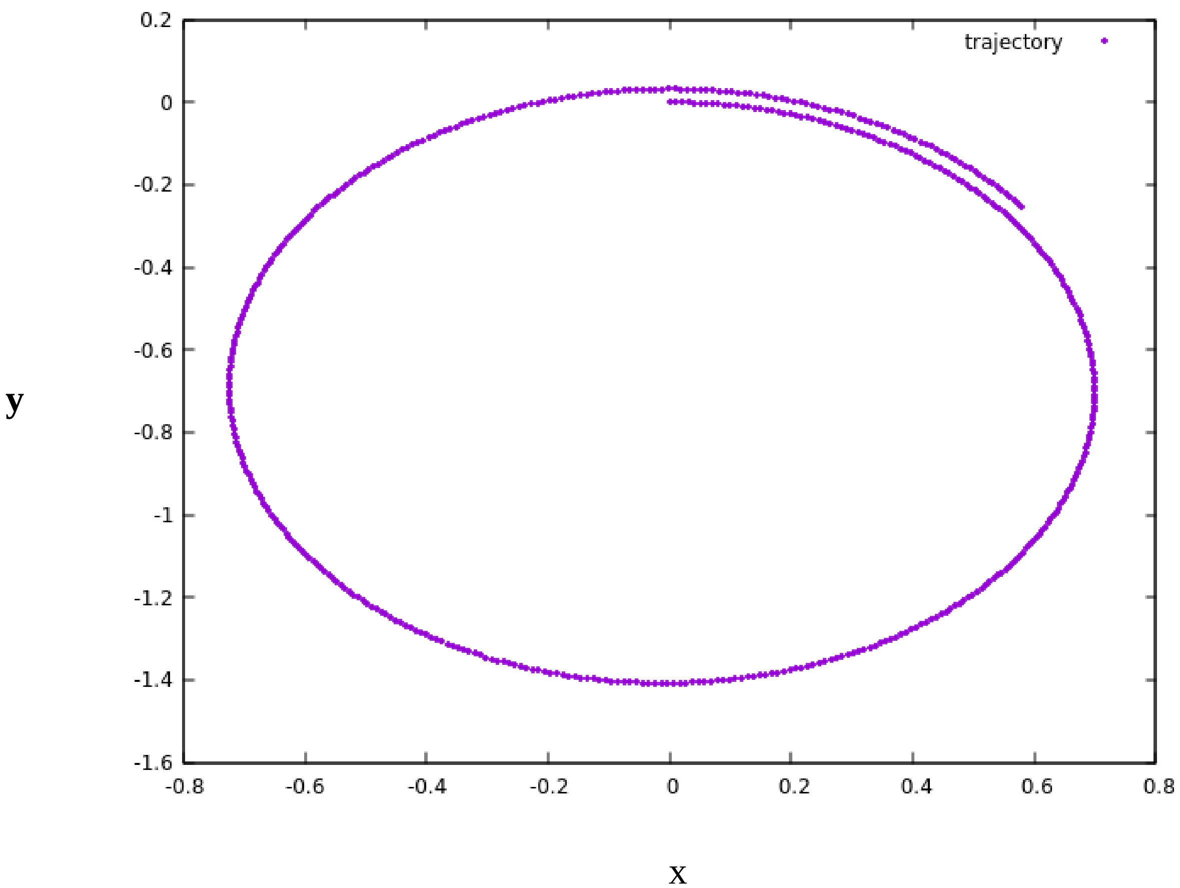
```
#include <stdio.h>
#include <math.h>
int main(void)
{const double B = 5.0E-9;
 const double q = 4.8E-18;
 const double m = 1.67E-27;
 const double V0 = 1.0E1;
 double Omega = q * B / m ;


 double x0, x1, y0, y1;
 double vx0, vx1, vy0, vy1;
   FILE *fp;
   fp=fopen("leafB0.dat","w");

 double dt = 0.001;
 int i;
 int n = 499;
   // initial conditions
         x0=0;
         y0=0;
         vx0=V0;
         vy0=0;
       fprintf(fp,"%f %f\n", x0, y0);
   // one step update of velocity
       vx0 = vx0- Omega * vy0 * 0.5 * dt;
       vy0 = vy0+ Omega * vx0 * 0.5 * dt;

      for(i = 0; i <= n; i++){
   // update the velocity
       vx1 =vx0+ Omega * vy0 * dt;
       vy1 =vy0- Omega * vx0 * dt;
   // update the position
       x1 = x0+ vx1 * dt;
       y1 =y0+ vy1 * dt;
   // write the position
       x0=x1;
       y0=y1;
       vx0=vx1;
       vy0=vy1;
       fprintf(fp,"%f %f\n", x0, y0)   }
   return 0;}
```

Important Note: In the above program, initial time is set to be zero, i.e., t_0 =0 and time step dt=0.001 and total number of steps n=499. Initial conditions are x_0 =0; v_{x0}=1, y_0 =0, v_{y0}=0. Total time=0+(499+1)(0.001)=0.5. When we run the program, an output file is also generated "leafB0.dat" in which position x and y are written. We can use this file to plot the trajectory of the particle to verify our numerical results with analytical expectation (which is circular trajectory for the chosen initial condition). Following plot shows the obtained trajectory from the Leap-frog method.



From the above plots, it appears that the trajectory otbained from the Leap-frog method is not perfectly closed circle (but is a spiral shape trajectory) which indicates that, due to the presence of numerical errors, the method is not perfectly suitable to study the motion of a charged particle in magnetic field.

<u>**Assignment**</u>

Write down and solve the equation of motion of a charged particle q moving in a uniform electric field **B=B**$_0$ $\hat{e}_z$ using the Leap-frog Method (***do it manually, not by C-programming***). Use following numerical values of the parameters in the calculations:

$B_0 = 5.0\text{x } 10^{-9}$;

$q = 4.8\text{x } 10^{-18}$;

$m = 1.67\text{x } 10^{-27}$;

$x_0 = y_0 = z_0 = 0$

$v_{x0} = 1,\ v_{y0} = 0,\ v_{z0} = 0$

$t_0 = 0$

$\Delta t = 0.1$

$n = 9$