Initial Value Problems (IVPs) & Boundary Value Problems (BVPs)

Course: MPPHYCC-05 Modeling and Simulation (M.Sc. II Sem)

Dr. Sanjay Kumar

Assistant Professor
Department of Physics
Patna University

Contact Details: Email- sainisanjay35@gmail.com;
Contact No.- 9413674416

Introductory Remarks on ODE and PDE

- A differential equation (DE) is simply an equation which involves one or more derivatives of an unknown function.

- The unknown function may depend on one or more independent variables.

- If the unknown is a function of a single variable then the derivatives occurring in the equation are ordinary derivatives and the equation is called an ordinary differential eqaution (ODE).

- If the unknown is a function of more than one independent variable then the derivatives are partial derivatives and the equation is called a partial differential equation (PDE).

Examples of ODEs

$$y'(t) = y^2(t)e^{-t}$$

$$-u''(x) + 2u(x) = 3\sin x\,.$$

Examples of PDEs

$$u_t(x,t) - u_{xx}(x,t) = f(x,t)$$

Initial Value Problems (IVPs)

- Following ODE can easily be solved anaytically $y'(t) = -\sin t + t$

$$\int y'(t)\, dt = \int (-\sin t + t)\, dt \Rightarrow y(t) + C_1 = \cos t + \frac{t^2}{2} + C_2 \Rightarrow y(t) = \cos t + \frac{t^2}{2} + C$$

- In order to uniquely determine y(t) we need to specify an auxiliary condition such as specifying y at some point. For example, if we specify y(0) = 0 then

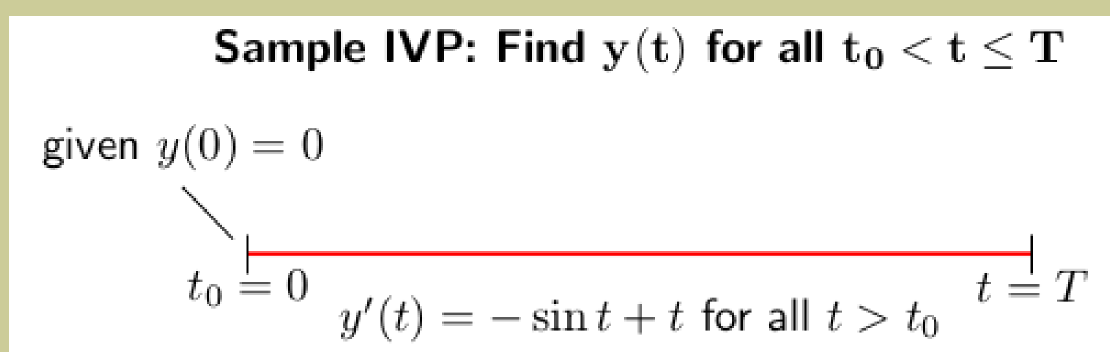
$$y(t) = \cos(t) + t^2/2 - 1.$$

- This auxiliary condition is called an initial condition and the problem

$$\begin{aligned} y'(t) &= -\sin t + t \quad 0 < t \le T \\ y(0) &= 0 \end{aligned}$$

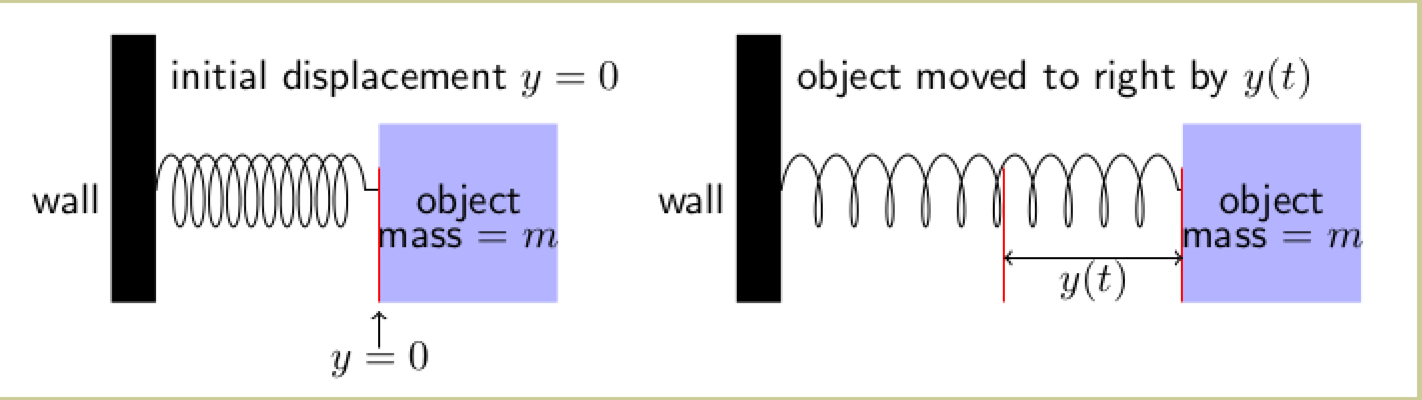
is called an initial value problem (IVP); here T denotes the final time.

- In general, for an IVP (with highest derivative one, i.e. first order ODE), we are given the unknown y at some time $t = t_0$ and are asked to determine y for subsequent times.



Initial Value Problems (Contd.)

- What about higher order ODE? Example $y''(t) = -\sin(t) + t$
- The general solution of this equation $y(t) = \sin t + t^3/6 + C_1 t + C_2$
- An example of an IVP for this equation is to specify both y and its derivative at t_0.
- For an IVP we must specify all auxiliary conditions at the same time t_0. We do not require $t_0 = 0$, but we often set it to zero for simplicity.
- Initial value problems can be either ordinary or partial differential equations.
- Example in Physics: Spring-mass system



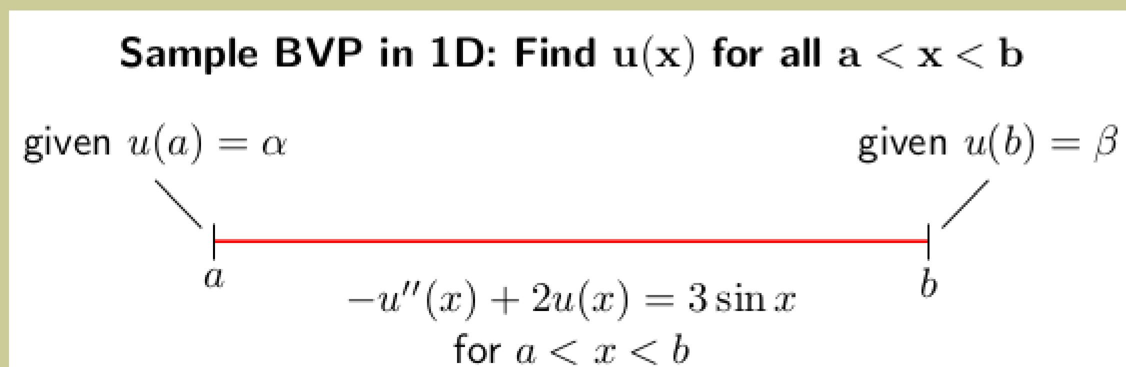
- ODEs is obtained from the Newton's second law $my''(t) = -ky(t) - cy'(t)$
- Need to specify two initial conditions at the same instance of time. Possible initial conditions can be: the initial displacement y(0)=0 and the initial velocity $y'(t=0)= v$.

Boundary Value Problems (BVPs)

- The differential equation models some phenomenon that does not depend on time such as a state of equilibrium. Example:

$$\begin{aligned} -u''(x) + 2u(x) &= 3\sin x \quad a < x < b \\ u(a) &= \alpha \\ u(b) &= \beta \end{aligned}$$

- Here specification of u at boundaries; i.e. u(a) and u(b) is known as boundary conditions (BCs) and the diffential equation with BCs is known as **boundary value problem (BVP)**.

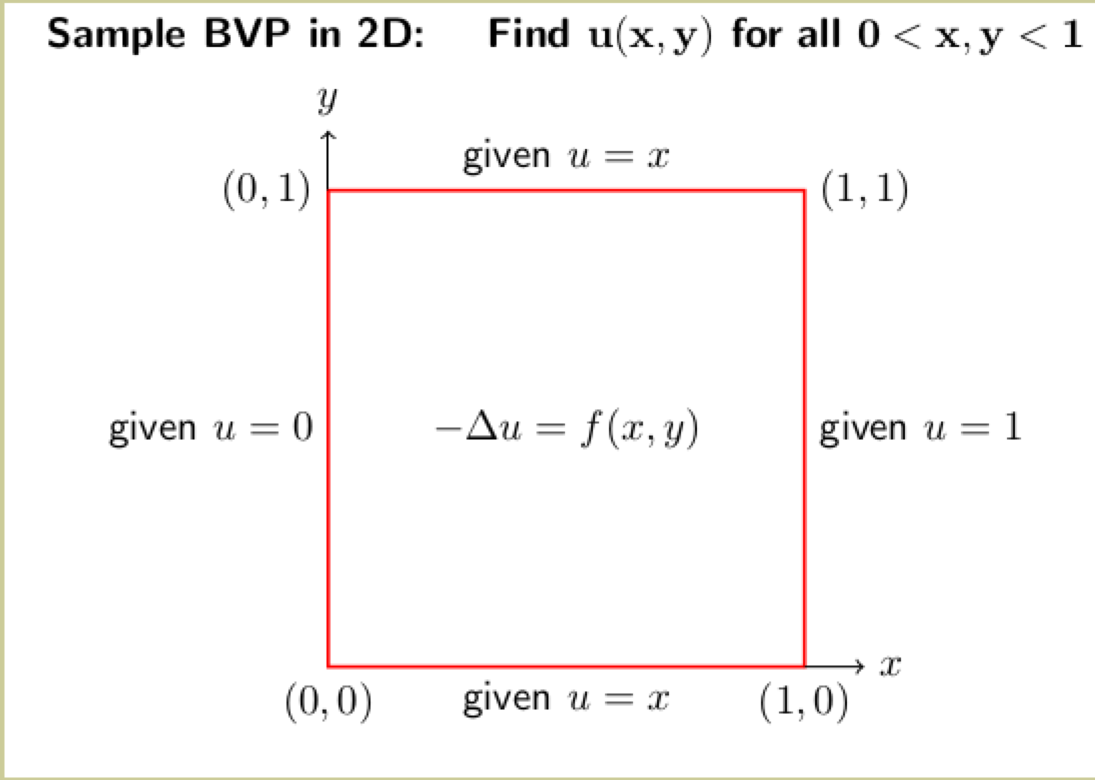


- Another example:

$$\begin{aligned} -\Delta u = -(u_{xx} + u_{yy}) &= f(x,y) \quad 0 < x < 1,\ 0 < y < 1 \\ u &= 0 \quad \text{for } x = 0 \qquad u = 1 \quad \text{for } x = 1 \\ u &= x \quad \text{for } y = 0, 1\,. \end{aligned}$$

Boundary Value Problems (Contd.)

Pictorial representation of the BVP



- There are other types of boundary conditions than specifying the unknown on the boundary. We can also specify the derivative of the unknown or a combination of the unknown and its derivative.

Difference between IVPs and BVPs

BVPs require auxiliary conditions to be imposed at the extremes of the domain (i.e., at the boundaries) whereas IVPs require auxiliary conditions to be imposed at the same point.

Assignments

In the following differential equations determine
(i) if each differential equation is an ODE or a PDE,
(ii) classify each problem as an IVP, a BVP or an IBVP
(iii) identify which conditions are initial conditions and which are boundary conditions.

1. Let u=u(x) then

$$u'''' - 2u'' + u = 0 \quad 0 < x < 1$$
$$u(0) = u(1) = 1, \quad u''(0) = 0, \quad u''(1) = -1$$

2. Let u=u(x) then

$$u'' = x^3 u^2 \quad 1 < x \leq 10$$
$$u(1) = 4, \quad u'(1) = 0$$

3. Let u=u(x,y) then

$$-(u_{xx} + u_{yy}) = f(x, y) \quad 0 < x < 1, 0 < y < 2$$
$$u(0, y) = u(1, y) = 1 \quad u_y(x, 0) = u_y(x, 2) = 0$$

4. Let u=u(x,t) then

$$u_t + uu_x - u_{xx} = f(x, t) \quad 0 < x < 2, 0 < t \leq 1$$
$$u(0, t) = \cos \pi t, \quad u(2, t) = 9$$
$$u(x, 0) = x^3 + 1\,.$$

Thanks for the attention!