# The Derivation of the Boltzann Equation

**Course: MPHYEC-01I Plasma Physics (M.Sc. IV Sem)**

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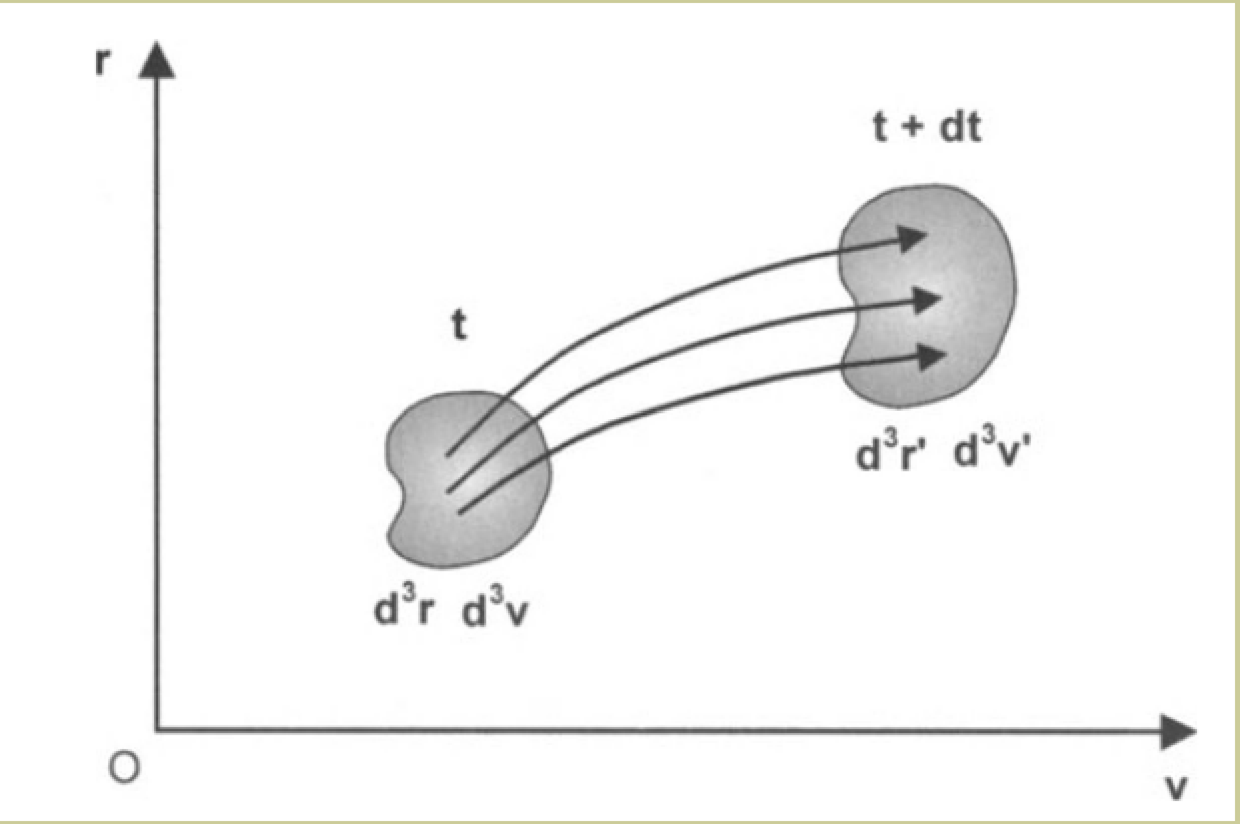
Lecture 2: Unit-IV

# Derivation of the collisionless Boltzmann Equation

Number of particles in a volume element in 6D phase space for the species α:

$$d^6\mathcal{N}_\alpha(\mathbf{r},\mathbf{v},t) = f_\alpha(\mathbf{r},\mathbf{v},t)\; d^3r\; d^3v$$

In presence of an external force, in absence of any collisions, the particles will move to another volume element after time dt.



Coordinates of volume element at t is related to volume element at t+dt as:

$$\mathbf{r}'(t+dt) = \mathbf{r}(t) + \mathbf{v}\ dt$$

$$\mathbf{v}'(t+dt) = \mathbf{v}(t) + \mathbf{a}\ dt$$

As in absence of collisions number of particles will be same in both the volume:

$$f_\alpha(\mathbf{r}', \mathbf{v}', t+dt)\ d^3r'\ d^3v' = f_\alpha(\mathbf{r}, \mathbf{v}, t)\ d^3r\ d^3v$$

$$d^3r'\ d^3v' = |J|\ d^3r\ d^3v$$

Jacobian of transformation

$$J = \frac{\partial(\mathbf{r}', \mathbf{v}')}{\partial(\mathbf{r}, \mathbf{v})} = \frac{\partial(x', y', z', v_x', v_y', v_z')}{\partial(x, y, z, v_x, v_y, v_z)}$$

$$J = \begin{pmatrix} \partial x'/\partial x & \partial y'/\partial x & \cdots & \partial v_z'/\partial x \\ \partial x'/\partial y & \partial y'/\partial y & \cdots & \partial v_z'/\partial y \\ \cdots & \cdots & \cdots & \cdots \\ \partial x'/\partial v_z & \partial y'/\partial v_z & \cdots & \partial v_z'/\partial v_z \end{pmatrix}$$

It can be shown that J=1

$$d^3r'\ d^3v' = d^3r\ d^3v$$

$$[f_\alpha(\mathbf{r}', \mathbf{v}', t+dt) - f_\alpha(\mathbf{r}, \mathbf{v}, t)]\ d^3r\ d^3v = 0$$

$$f_\alpha(\mathbf{r} + \mathbf{v}\ dt, \mathbf{v} + \mathbf{a}\ dt, t+dt) = f_\alpha(\mathbf{r}, \mathbf{v}, t) + \left[\ \frac{\partial f_\alpha}{\partial t} + \right.$$

$$\left.\left(v_x\frac{\partial f_\alpha}{\partial x} + v_y\frac{\partial f_\alpha}{\partial y} + v_z\frac{\partial f_\alpha}{\partial z}\right) + \left(a_x\frac{\partial f_\alpha}{\partial v_x} + a_y\frac{\partial f_\alpha}{\partial v_y} + a_z\frac{\partial f_\alpha}{\partial v_z}\right)\right]\ dt$$

$$\nabla = \widehat{\mathbf{x}}\frac{\partial}{\partial x} + \widehat{\mathbf{y}}\frac{\partial}{\partial y} + \widehat{\mathbf{z}}\frac{\partial}{\partial z}$$

$$\nabla_v = \widehat{\mathbf{x}}\frac{\partial}{\partial v_x} + \widehat{\mathbf{y}}\frac{\partial}{\partial v_y} + \widehat{\mathbf{z}}\frac{\partial}{\partial v_z}$$

$$f_\alpha(\mathbf{r} + \mathbf{v}\ dt, \mathbf{v} + \mathbf{a}\ dt, t+dt) = f_\alpha(\mathbf{r}, \mathbf{v}, t) +$$

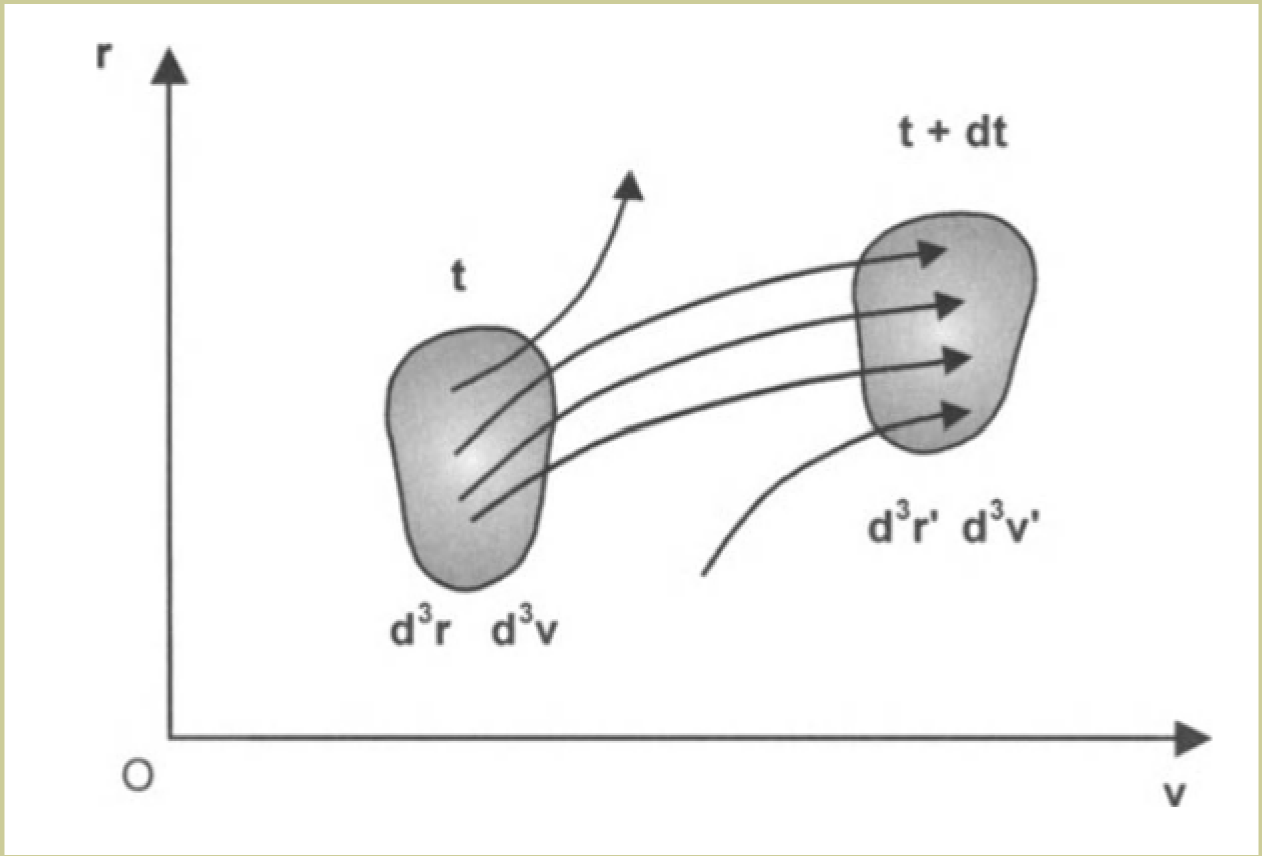
$$\left[\frac{\partial f_\alpha(\mathbf{r}, \mathbf{v}, t)}{\partial t} + \mathbf{v}\cdot\nabla f_\alpha(\mathbf{r}, \mathbf{v}, t) + \mathbf{a}\cdot\nabla_v f_\alpha(\mathbf{r}, \mathbf{v}, t)\right]\ dt$$

$$\frac{\partial f_\alpha(\mathbf{r},\mathbf{v},t)}{\partial t} + \mathbf{v}\cdot\nabla f_\alpha(\mathbf{r},\mathbf{v},t) + \mathbf{a}\cdot\nabla_v f_\alpha(\mathbf{r},\mathbf{v},t) = 0$$

$$\frac{\mathcal{D} f_\alpha(\mathbf{r},\mathbf{v},t)}{\mathcal{D} t} = 0$$

**The Boltzmann equation for collisionless system.**

## Effect of collisions



$$[f_\alpha(\mathbf{r}', \mathbf{v}', t + dt) - f_\alpha(\mathbf{r}, \mathbf{v}, t)]\ d^3r\ d^3v = \left(\frac{\delta f_\alpha}{\delta t}\right)_{coll} d^3r\ d^3v\ dt$$

$$\frac{\partial f_\alpha}{\partial t} + \mathbf{v} \cdot \nabla f_\alpha + \mathbf{a} \cdot \nabla_v f_\alpha = \left(\frac{\delta f_\alpha}{\delta t}\right)_{coll}$$

**The Boltzmann equation for collisional system.**

## Relaxation model for collisional term

**Assumption:** The effect of collisions is to restore a situation of local equilibrium, characterized by a local equilibrium distribution function $f_{\alpha 0}(\mathbf{r}, \mathbf{v})$. In the absence of external forces, an initial non-equilibrium state is assumed which is described by a distribution function $f_\alpha(\mathbf{r}, \mathbf{v}, t)$ different from $f_{\alpha 0}(\mathbf{r}, \mathbf{v})$, reaches a local equilibrium condition exponentially with time due to collisions.

$$\left(\frac{\delta f_\alpha}{\delta t}\right)_{coll} = -\frac{(f_\alpha - f_{\alpha 0})}{\tau}$$

τ represents the relaxation time.

$$\frac{\partial f_\alpha}{\partial t} = -\frac{(f_\alpha - f_{\alpha 0})}{\tau}$$

In absence of external force and Spatial gradients.

$$\frac{\partial f_\alpha}{\partial t} + \frac{f_\alpha}{\tau} = \frac{f_{\alpha 0}}{\tau}$$

$$f_\alpha(\mathbf{v}, t) = f_{\alpha 0} + [f_\alpha(\mathbf{v}, 0) - f_{\alpha 0}]\ e^{-t/\tau}$$

***Thanks for the attention!***