Transformations

Transformations are used to ***scale, translate, rotate, reflect*** and ***shear*** shapes and objects. And, as we shall discover shortly, it is possible to affect this by changing their coordinate values. Although algebra is the basic notation for transformations, it is also possible to express them into *matrices*.

Translation

Cartesian coordinates provide a one-to-one relationship between number and shape, such that when we change a shape's coordinates, we change its geometry. For example, if *P* (*x, y*) is a vertex on a shape, when we apply the operation *x* = *x* + 3 we create a new point *P* (*x* , *y*) three units to the right. Similarly, the operation *y* = *y* + 1 creates a new point *P* (*x, y*) displaced one unit vertically. By applying both of these transforms to every vertex to the original shape, the shape is displaced as shown in *Figure.*

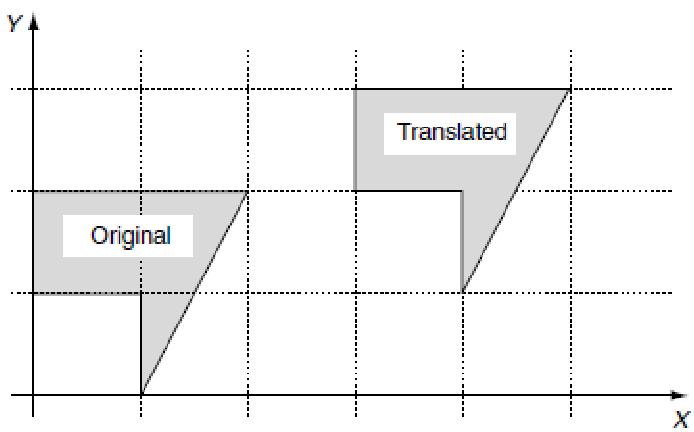

Figure The translated shape results by adding 3 to every *x*-coordinate, and 1 to every *y*- coordinate of the original shape.

Scaling

Shape scaling is achieved by multiplying coordinates as follows:

$$x' = 2x$$
$$y' = 1.5y$$

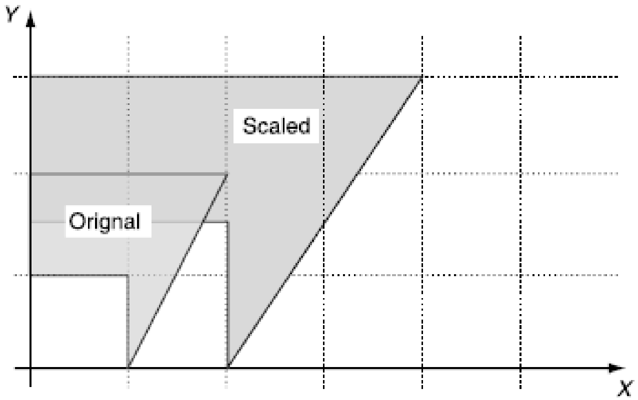

Figure The scaled shape results by multiplying every *x*-coordinate by 2 and every *y*-coordinate by 1.5.

This transform results in a horizontal scaling of 2 and a vertical scaling of 1.5, as illustrated in *Figure 3.2*. Note that a point located at the origin does not change its place, so scaling is relative to the origin.

Reflection

To make a reflection of a shape relative to the *y* -axis, we simply reverse the sign of the
x -coordinate, leaving the *y* -coordinate unchanged

$$x' = -x$$
$$y' = y$$

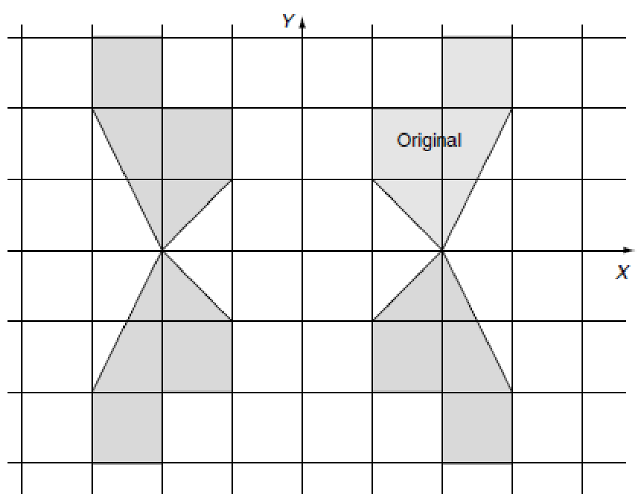

Figure The top right-hand shape can give rise to the three reflections simply by reversing the signs of coordinates.

and to reflect a shape relative to the *x* -axis we reverse the *y* -coordinates:

$$x' = x$$
$$y' = -y$$

2D Transformations

$$x' = x + t_x$$
$$y' = y + t_y$$

The algebraic and matrix notation for 2D translation is
or, using matrices,

$$\begin{bmatrix} x' \\ y' \\ 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & t_x \\ 0 & 1 & t_y \\ 0 & 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} x \\ y \\ 1 \end{bmatrix}$$

2D Scaling

The algebraic and matrix notation for 2D scaling is

$$x' = s_x x$$
$$y' = s_y y$$

or, using matrices,

$$\begin{bmatrix} x' \\ y' \\ 1 \end{bmatrix} = \begin{bmatrix} s_x & 0 & 0 \\ 0 & s_y & 0 \\ 0 & 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} x \\ y \\ 1 \end{bmatrix}$$

The scaling action is relative to the origin, i.e. the point (0,0) remains (0,0) All other points move away from the origin. To scale relative to another point (p_x, p_y) we first subtract (p_x, p_y) from (*x*, *y*) respectively. This effectively translates the reference point (p_x, p_y) back to the origin. Second, we perform the scaling operation, and third, add (p_x, p_y) back to (*x*, *y*) respectively, to compensate for the original subtraction. Algebraically this is

$$x' = s_x(x - p_x) + p_x$$
$$y' = s_y(y - p_y) + p_y$$

which simplifies to

$$x' = s_x x + p_x(1 - s_x)$$
$$y' = s_y y + p_y(1 - s_y)$$

or in a homogeneous matrix form

$$\begin{bmatrix} x' \\ y' \\ 1 \end{bmatrix} = \begin{bmatrix} s_x & 0 & p_x(1-s_x) \\ 0 & s_y & p_y(1-s_y) \\ 0 & 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} x \\ y \\ 1 \end{bmatrix} \qquad (1)$$

For example, to scale a shape by 2 relative to the point (1, 1) the matrix is

$$\begin{bmatrix} x' \\ y' \\ 1 \end{bmatrix} = \begin{bmatrix} 2 & 0 & -1 \\ 0 & 2 & -1 \\ 0 & 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} x \\ y \\ 1 \end{bmatrix}$$

2D Reflections

The matrix notation for reflecting about the *y* -axis is:

$$\begin{bmatrix} x' \\ y' \\ 1 \end{bmatrix} = \begin{bmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} x \\ y \\ 1 \end{bmatrix}$$

or about the *x* -axis

$$\begin{bmatrix} x' \\ y' \\ 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} x \\ y \\ 1 \end{bmatrix}$$

However, to make a reflection about an arbitrary vertical or horizontal axis we need to introduce some more algebraic deception. For example, to make a reflection about the vertical axis $x = 1$, we first subtract 1 from the *x*-coordinate. This effectively makes the $x = 1$ axis coincident with the major *y* -axis. Next we perform the reflection by reversing the sign of the modified *x*-coordinate. And finally, we add 1 to the reflected coordinate to compensate for the original subtraction. Algebraically, the three steps are

$$x_1 = x - 1$$
$$x_2 = -(x - 1)$$
$$x' = -(x - 1) + 1$$

which simplifies to

$$x' = -x + 2$$
$$y' = y$$

or in matrix form,

$$\begin{bmatrix} x' \\ y' \\ 1 \end{bmatrix} = \begin{bmatrix} -1 & 0 & 2 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} x \\ y \\ 1 \end{bmatrix}$$

Figure illustrates this process.

In general, to reflect a shape about an arbitrary *y*-axis, *y*=a_x , the following transform is required:

$$x' = -(x - a_x) + a_x \quad = -x + 2a_x$$
$$y' = y$$

or, in matrix form,

$$\begin{bmatrix} x' \\ y' \\ 1 \end{bmatrix} = \begin{bmatrix} -1 & 0 & 2a_x \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} x \\ y \\ 1 \end{bmatrix} \qquad (2)$$

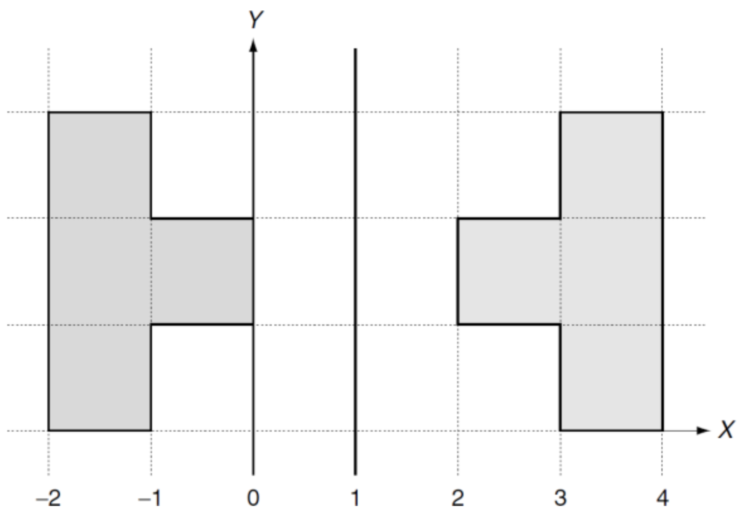

Figure The shape on the right is reflected about the *x* = 1 axis.

Similarly, this transform is used for reflections about an arbitrary *x* -axis, $y = a_y$:

$$x' = x$$
$$y' = -(y - a_y) + a_y = -y + 2a_y$$

or, in matrix form,

$$\begin{bmatrix} x' \\ y' \\ 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 2a_y \\ 0 & 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} x \\ y \\ 1 \end{bmatrix}$$

2D Shearing

A shape is sheared by leaning it over at an angle *β*. *Figure 3.5* illustrates the geometry, and we see that the *y* -coordinate remains unchanged but the *x* -coordinate is a function of *y* and tan(*β*).

$$x' = x + y\tan(\beta)$$
$$y' = y$$

or, in matrix form,

$$\begin{bmatrix} x' \\ y' \\ 1 \end{bmatrix} = \begin{bmatrix} 1 & \tan(\beta) & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} x \\ y \\ 1 \end{bmatrix}$$

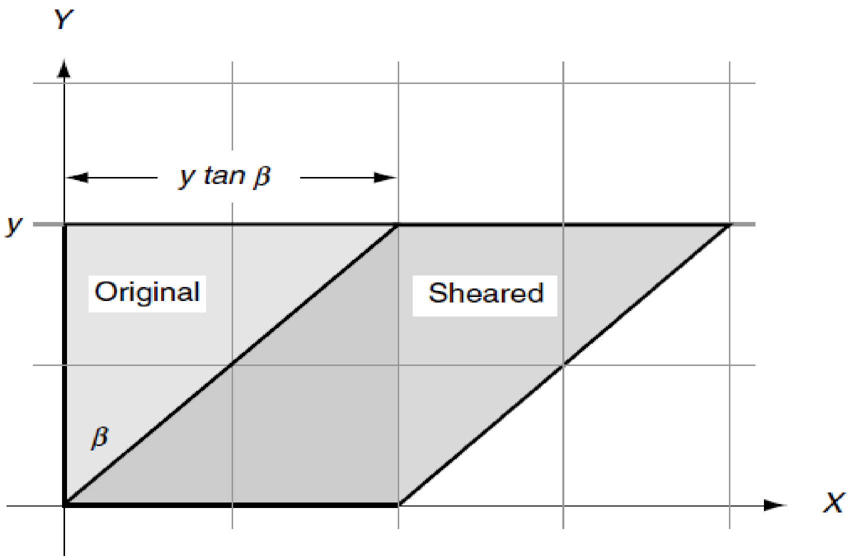

Figure The original square shape is sheared to the right by an angle *β*, and the horizontal shift is proportional to *ytan*(*β*).

2D Rotation

Figure: shows a point *P* (*x*, *y*) which is to be rotated by an angle *β* about the origin to

$$x' = R\cos(\theta + \beta)$$
$$y' = R\sin(\theta + \beta)$$

P′(*x*′ , *y*′). It can be seen that

therefore,

$$x' = R(\cos(\theta)\cos(\beta) - \sin(\theta)\sin(\beta))$$
$$y' = R(\sin(\theta)\cos(\beta) + \cos(\theta)\sin(\beta))$$
$$x' = R\left(\frac{x}{R}\cos(\beta) - \frac{y}{R}\sin(\beta)\right)$$
$$y' = R\left(\frac{y}{R}\cos(\beta) + \frac{x}{R}\sin(\beta)\right)$$
$$x' = x\cos(\beta) - y\sin(\beta)$$
$$y' = x\sin(\beta) + y\cos(\beta)$$

or, in matrix form,

$$\begin{bmatrix} x' \\ y' \\ 1 \end{bmatrix} = \begin{bmatrix} \cos(\beta) & -\sin(\beta) & 0 \\ \sin(\beta) & \cos(\beta) & 0 \\ 0 & 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} x \\ y \\ 1 \end{bmatrix}$$

For example, to rotate a point by 90° the matrix becomes

$$\begin{bmatrix} x' \\ y' \\ 1 \end{bmatrix} = \begin{bmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} x \\ y \\ 1 \end{bmatrix}$$

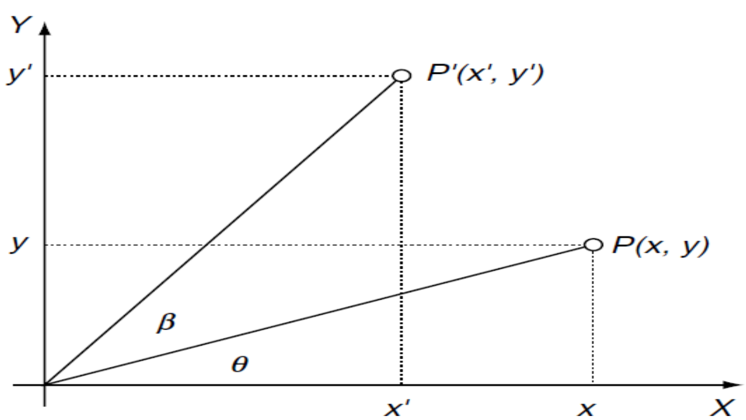

Figure The point *P* (*x, y*) is rotated through an angle *β* to *P* (*x* , *y*).

Thus the point (1, 0) becomes (0, 1). If we rotate by 360° the matrix becomes

$$\begin{bmatrix} x' \\ y' \\ 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} x \\ y \\ 1 \end{bmatrix}$$

Such a matrix has a null effect and is called an *identity matrix*.