3D Transformations

Now we come to transformations in **three dimensions**, where we apply the same reasoning as in two dimensions. Scaling and translation are basically the same, but where in 2D we rotated a shape about *a point*, in 3D we rotate an object about *an axis*.

3D Translation

The algebra is so simple for 3D translation that we can write the homogeneous matrix directly:

$$\begin{bmatrix} x' \\ y' \\ z' \\ 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & t_x \\ 0 & 1 & 0 & t_y \\ 0 & 0 & 1 & t_z \\ 0 & 0 & 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} x \\ y \\ z \\ 1 \end{bmatrix}$$

3D Scaling

The algebra for 3D scaling is

$$x' = s_x x$$
$$y' = s_y y$$
$$z' = s_z z$$

which in matrix form is

$$\begin{bmatrix} x' \\ y' \\ z' \\ 1 \end{bmatrix} = \begin{bmatrix} s_x & 0 & 0 & 0 \\ 0 & s_y & 0 & 0 \\ 0 & 0 & s_z & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} x \\ y \\ z \\ 1 \end{bmatrix}$$

The scaling is relative to the origin, but we can arrange for it to be relative to an arbitrary point (p_x, p_y , p_z) with the following algebra:

$$x' = s_x(x - p_x) + p_x$$
$$y' = s_y(y - p_y) + p_y$$
$$z' = s_z(z - p_z) + p_z$$

which in matrix form is

$$\begin{bmatrix} x' \\ y' \\ z' \\ 1 \end{bmatrix} = \begin{bmatrix} s_x & 0 & 0 & p_x(1 - s_x) \\ 0 & s_y & 0 & p_y(1 - s_y) \\ 0 & 0 & s_z & p_z(1 - s_z) \\ 0 & 0 & 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} x \\ y \\ z \\ 1 \end{bmatrix}$$

3D Rotations

In two dimensions a shape is rotated about a point, whether it is the origin or some arbitrary position. In three dimensions an object is rotated about an axis, whether it is the x -, y - or z - axis, or some arbitrary axis. To begin with, let's look at rotating a vertex about one of the three orthogonal axes; such rotations are called *Euler rotations* after the Swiss mathematician Leonhard Euler (1707–1783).

Recall that a general 2D-rotation transform is given by

$$\begin{bmatrix} x' \\ y' \\ 1 \end{bmatrix} = \begin{bmatrix} \cos(\beta) & -\sin(\beta) & 0 \\ \sin(\beta) & \cos(\beta) & 0 \\ 0 & 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} x \\ y \\ 1 \end{bmatrix}$$

which in 3D can be visualized as rotating a point P (x, y, z) on a plane parallel with the xy - plane as shown in *Figure 3.7*. In algebraic terms this can be written as

$$x' = x\cos(\beta) - y\sin(\beta)$$
$$y' = x\sin(\beta) + y\cos(\beta)$$
$$z' = z$$

Therefore, the 3D transform can be written as

$$\begin{bmatrix} x' \\ y' \\ z' \\ 1 \end{bmatrix} = \begin{bmatrix} \cos(\beta) & -\sin(\beta) & 0 & 0 \\ \sin(\beta) & \cos(\beta) & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} x \\ y \\ z \\ 1 \end{bmatrix}$$

which basically rotates a point about the z -axis.

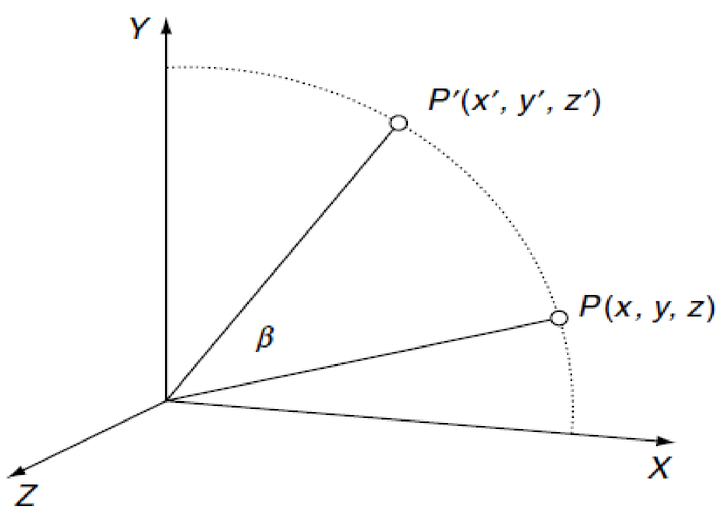


Figure Rotating P about the z-axis.

When rotating about the *x -axis*, the *x* -coordinate remains constant while the *y* - and *z* -coordinates are changed. Algebraically, this is

$$
\begin{aligned}
x' &= x \\
y' &= y\cos(\beta) - z\sin(\beta) \\
z' &= y\sin(\beta) + z\cos(\beta)
\end{aligned}
$$

or, in matrix form

$$
\begin{bmatrix} x' \\ y' \\ z' \\ 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & \cos(\beta) & -\sin(\beta) & 0 \\ 0 & \sin(\beta) & \cos(\beta) & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} x \\ y \\ z \\ 1 \end{bmatrix}
$$

When rotating about the *y -axis*, the *y* -coordinate remains constant while the *x* - and *z* -coordinates are changed. Algebraically, this is

$$
\begin{aligned}
x' &= z\sin(\beta) + x\cos(\beta) \\
y' &= y \\
z' &= z\cos(\beta) - x\sin(\beta)
\end{aligned}
$$

or, in matrix form

$$
\begin{bmatrix} x' \\ y' \\ z' \\ 1 \end{bmatrix} = \begin{bmatrix} \cos(\beta) & 0 & \sin(\beta) & 0 \\ 0 & 1 & 0 & 0 \\ -\sin(\beta) & 0 & \cos(\beta) & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} x \\ y \\ z \\ 1 \end{bmatrix}
$$

Note that the matrix terms do not appear to share the symmetry seen in the previous two matrices. Nothing has really gone wrong, it is just the way the axes are paired together to rotate the coordinates.

The above rotations are also known as ***yaw***, ***pitch*** and ***roll***. Great care should be taken with these terms when referring to other books and technical papers. Sometimes a left-handed system of axes is used rather than a right-handed set, and the vertical axis may be the *y* -axis or the *z* -axis.

Consequently, the matrices representing the rotations can vary greatly. In this text all Cartesian coordinate systems are right-handed, and the vertical axis is always the *y* -axis. The roll, pitch and yaw angles can be defined as follows:

- ***roll*** is the angle of rotation about the ***z* -axis**

- ***pitch*** is the angle of rotation about the ***x* -axis**
- ***yaw*** is the angle of rotation about the ***y* -axis**

Figure: illustrates these rotations and the sign convention. The homogeneous matrices representing these rotations are as follows:

- rotate *roll* about the *z* -axis:

$$\begin{bmatrix} \cos(roll) & -\sin(roll) & 0 & 0 \\ \sin(roll) & \cos(roll) & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

- rotate *pitch* about the *x* -axis:

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & \cos(pitch) & -\sin(pitch) & 0 \\ 0 & \sin(pitch) & \cos(pitch) & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

- rotate *yaw* about the *y* -axis:

$$\begin{bmatrix} \cos(yaw) & 0 & \sin(yaw) & 0 \\ 0 & 1 & 0 & 0 \\ -\sin(yaw) & 0 & \cos(yaw) & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

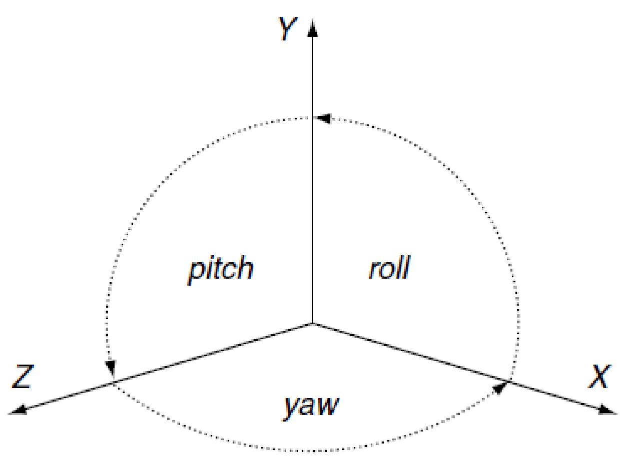


Figure The convention for *roll*, *pitch* and *yaw* angles.

A common sequence for applying these rotations is *roll, pitch, yaw*, as seen in the following transform:

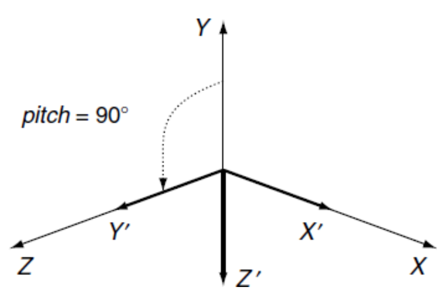
$$\begin{bmatrix} x' \\ y' \\ z' \\ 1 \end{bmatrix} = [yaw] \cdot [pitch] \cdot [roll] \cdot \begin{bmatrix} x \\ y \\ z \\ 1 \end{bmatrix}$$

and if a translation is involved,

$$\begin{bmatrix} x' \\ y' \\ z' \\ 1 \end{bmatrix} = [translate] \cdot [yaw] \cdot [pitch] \cdot [roll] \cdot \begin{bmatrix} x \\ y \\ z \\ 1 \end{bmatrix}$$

When these rotation transforms are applied, the vertex is first rotated about the *z* -axis (roll), followed by a rotation about the *x* -axis (pitch), followed by a rotation about the *y* -axis (yaw). Euler rotations are relative to the fixed frame of reference. This is not always easy to visualize, as one's attention is normally with the rotating frame of reference.

Let's consider a simple example where an axial system is subjected to a pitch rotation followed by a yaw rotation relative to fixed frame of reference. We begin with two frames of reference ***XYZ*** and ***X' Y' Z'*** mutually aligned. *Figure 3.9* shows the orientation of ***X' Y' Z'*** after it is subjected to a pitch of 90° about the *x* -axis. *Figure 3.10* shows the final orientation after ***X' Y' Z'*** is subjected to a yaw of 90° about the *y* -axis.



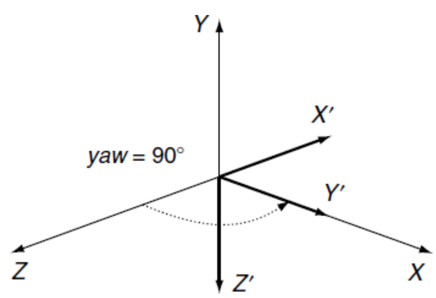


Fig. The *X' Y' Z'* axial system after a *pitch* of 90°. **Fig.** The *X' Y' Z'* axial system after a *yaw* of 90°.

Rotating about an Axis

The above rotations were relative to the *x* -, *y* - and *z* -axes. Now let's consider rotations about an axis parallel to one of these axes. To begin with, we will rotate about an axis parallel with the *z* -axis, as shown in *Figure*.

The scenario is very reminiscent of the 2D case for rotating a point about an arbitrary point, and the general transform is given by

$$\begin{bmatrix} x' \\ y' \\ z' \\ 1 \end{bmatrix} = [translate(p_x, p_y, 0)].[rotate\beta].[translate(-p_x, -p_y, 0)].\begin{bmatrix} x \\ y \\ z \\ 1 \end{bmatrix}$$

and the matrix is

$$\begin{bmatrix} x' \\ y' \\ z' \\ 1 \end{bmatrix} = \begin{bmatrix} \cos(\beta) & -\sin(\beta) & 0 & p_x(1-\cos(\beta)) + p_y\sin(\beta) \\ \sin(\beta) & \cos(\beta) & 0 & p_y(1-\cos(\beta)) - p_x\sin(\beta) \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} . \begin{bmatrix} x \\ y \\ z \\ 1 \end{bmatrix}$$

I hope you can see the similarity between rotating in 3D and 2D: the *x* - and *y* -coordinates are updated while the *z* -coordinate is held constant.

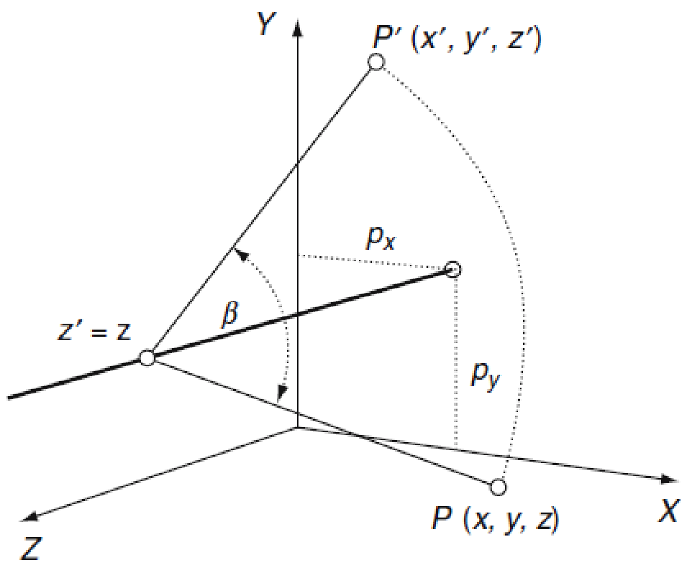


Figure : Rotating a point about an axis parallel with the *z*-axis

We can now state the other two matrices for rotating about an axis parallel with the *x* -axis and parallel with the *y* -axis:

- rotating about an axis parallel with the *x* -axis:

$$\begin{bmatrix} x' \\ y' \\ z' \\ 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & \cos(\beta) & -\sin(\beta) & p_y(1-\cos(\beta)) + p_z\sin(\beta) \\ 0 & \sin(\beta) & \cos(\beta) & p_z(1-\cos(\beta)) - p_y\sin(\beta) \\ 0 & 0 & 0 & 1 \end{bmatrix} . \begin{bmatrix} x \\ y \\ z \\ 1 \end{bmatrix}$$

- rotating about an axis parallel with the *y* -axis:

$$\begin{bmatrix} x' \\ y' \\ z' \\ 1 \end{bmatrix} = \begin{bmatrix} \cos(\beta) & 0 & \sin(\beta) & p_x(1-\cos(\beta)) - p_z\sin(\beta) \\ 0 & 1 & 0 & 0 \\ -\sin(\beta) & 0 & \cos(\beta) & p_z(1-\cos(\beta)) + p_x\sin(\beta) \\ 0 & 0 & 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} x \\ y \\ z \\ 1 \end{bmatrix}$$

3D Reflections

Reflections in 3D occur with respect to a plane, rather than an axis. The matrix giving the reflection relative to the *yz* -plane is

$$\begin{bmatrix} x' \\ y' \\ z' \\ 1 \end{bmatrix} = \begin{bmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} x \\ y \\ z \\ 1 \end{bmatrix}$$

and the reflection relative to a plane parallel to, and a_x units from, the *yz*- plane is

$$\begin{bmatrix} x' \\ y' \\ z' \\ 1 \end{bmatrix} = \begin{bmatrix} -1 & 0 & 0 & 2a_x \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} x \\ y \\ z \\ 1 \end{bmatrix}$$