Mathematical Expectation

MATHEMATICAL EXPECTATION

The expected value of the random variable X or the expectation of the random variable X, denoted by $E(X)$ is defined by

$$E(X) = \begin{cases} \displaystyle\int x\, f_X(x)dx, & \text{if } x \text{ is a cont. r.v.,} \\ \displaystyle\sum x\, p(x), & \text{if } x \text{ is a disc. r.v..} \end{cases}$$

Ex. What is expected value of the number of points obtained in a single throw with an ordinary dice?

Sol. Here the random variable X is the number of points of the dice which assumes the values 1,2,3,4,5,6 each with a probability $\frac{1}{6}$

$X = x$	0	1	2	3	4	5	6
$P(X = x)$	$\frac{1}{6}$	$\frac{1}{6}$	$\frac{1}{6}$	$\frac{1}{6}$	$\frac{1}{6}$	$\frac{1}{6}$	$\frac{1}{6}$

$E(X) = \sum x\, p(x) = 1 \times \frac{1}{6} + 2 \times \frac{1}{6} + 3 \times \frac{1}{6} + 4 \times \frac{1}{6} + 5 \times \frac{1}{6} + 6 \times \frac{1}{6}$

$= (1 + 2 + 3 + 4 + 5 + 6) \times \frac{1}{6} = \frac{21}{6}$

Ex. Suppose the random variable X takes the values 0,1,2,...with probability mass function

$$p(x) = \frac{e^{-\lambda}\, \lambda^x}{x!}$$

Sol. We know that $E(X) = \displaystyle\sum_{x=0}^{\infty} x\, p(x) = \sum_{x=0}^{\infty} x\, \frac{e^{-\lambda}\, \lambda^x}{x!}$

$= \lambda\, e^{-\lambda} \displaystyle\sum_{x=1}^{\infty} \frac{\lambda^{(x-1)}}{(x-1)!} = \lambda\, e^{-\lambda}\; e^{\lambda} = \lambda$

Ex. Suppose the random variable X with probability density function $f(x) = \lambda\, e^{-\lambda\, x}, \quad x > 0$

Sol. We know that $E(X) = \displaystyle\int_0^{\infty} x\, f(x)dx = \int_0^{\infty} x\, \lambda\, e^{-\lambda\, x} dx =$

$= \lambda \displaystyle\int_0^{\infty} x\, e^{-\lambda\, x} dx = \lambda \int_0^{\infty} x^{(2-1)}\, e^{-\lambda\, x} dx = \frac{\lambda}{\lambda^2} = \frac{1}{\lambda}$

Th. If a and b are constants, then $E(a\, X + b) = a\, E(X) + b$

Prof. $E(X) = \sum (a\,x + b)\,p(x) = a\sum x\,p(x) + b\sum p(x) = a\,E(X) + b.$

Th. The expectation of the sum of two random variable X and Y is equal to the sum of their expectation.

$$E(X + Y) = E(X) + E(Y).$$

Prof. $E(X+Y) = \int\int (x+y) f(x,\,y) dx\,dy$

$$= \int\int x\,f(x,\,y)dx\,dy + \int\int y\,f(x,\,y)dx\,dy$$

$$= \int x \int f(x,\,y)dy\,dx + \int y \int f(x,\,y)dx\,dy$$

$$= \int x\,g(x)\,dx + \int y\,h(y)\,dy = E(X) + E(Y)$$

so, $E(X+Y) = E(X) + E(Y)$.

Th. If X and Y be two independent random variable, then

$$E(X\,Y) = E(X)\,E(Y).$$

Prof. Since X and Y be two independent random variable, then $f(x,\,y) = f(x)\,f(y)$

$$E(X\,Y) = \int\int x\,y\,f(x,\,y)dx\,dy = \int\int x\,y\,f(x)\,f(y)dx\,dy$$

$$= \int x\,f(x)dx \int y\,f(y)dy = E(X)\,E(Y).$$

so $E(X\,Y) = E(X)\,E(Y)$.

Expectation of a Linear Combination of Random Variables : Suppose $Y_1, Y_2, ..., Y_m$ be any m random variables and if $b_1, b_2, ..., b_m$ are any m constants, then

$$E\Big(\sum_{j=1}^{m} b_j\,Y_j\Big) = \sum_{j=1}^{m} b_j E(Y_j)$$

MATHEMATICAL EXPECTATION of a Function of a Random Variable

The expected value of a function of a random variable denoted by $E(h(X))$ is defined by

$$E(h(X)) = \begin{cases} \int h(x)\,f_X(x)dx, & \text{if } h(x) \text{ is a cont. r.v.,} \\ \sum h(x)\,p(x), & \text{if } h(x) \text{ is a disc. r.v..} \end{cases}$$

VARIANCE

The variance of the random variable X or the variance of the random variable X, denoted by $V(X)$ is defined by

$$V(X) = \mu_2 = E(x-\bar{x})^2 = \begin{cases} \displaystyle\int (x-\bar{x})^2 \, f_X(x) dx, & \text{if } x \text{ is a cont. r.v.,} \\ \sum (x-\bar{x})^2 \, p(x), & \text{if } x \text{ is a disc. r.v..} \end{cases}$$

where, $\bar{x} = \mu = E(X)$

Th. $V(X) = E(X^2) - \Big(E(X)\Big)^2$

Prof. $V(X) = E(X-\bar{X})^2 = E(X^2 + \bar{X}^2 - 2\,\bar{X}\,X)$

$= E(X^2) + \bar{X}^2 - 2\,\bar{X}\,E(X) = E(X^2) + \bar{X}^2 - 2\,\bar{X}\,\bar{X} = E(X^2) - \bar{X}^2$

So, $V(X) = E(X^2) - \Big(E(X)\Big)^2$

Th. $V(a\,X+b) = a^2\,V(X)$

Prof. $Y = a\,X + b$ then $E(Y) = a\,E(X) + b$

$\Rightarrow Y - E(Y) = a(X - E(X)),$

Squaring and taking expectation of both sides, we get

$E(Y-E(Y))^2 = a^2\,E(X-E(X))^2 \Rightarrow V(Y) = a^2\,V(X)$

$\Rightarrow V(a\,X+b) = a^2\,V(X)$

Th. $V(X+Y) = V(X) + V(Y) + 2\,Cov(X,\,Y)$

Prof. $E(X+Y) = E(X) + E(Y)$ then

$\Rightarrow V(X+Y) = E\Big[(X+Y) - E(X+Y)\Big]^2$

$\Rightarrow E\Big[(X+Y) - E(X) - E(Y)\Big]^2$

$\Rightarrow E\Big[(X-E(X)) + (Y-E(Y))\Big]^2$

$\Rightarrow E\Big[(X-E(X))^2 + (Y-E(Y))^2 + 2\,(X-E(X))\,(Y-E(Y))\Big]$

$\Rightarrow E(X-E(X))^2+E(Y-E(Y))^2+2\,E((X-E(X))\,(Y-E(Y)))$

$= V(X)+V(Y)+2\,Cov(X,\,Y)$

So, $V(X+Y)=V(X)+V(Y)+2\,Cov(X,\,Y)$

Th. $Cov(X\,,Y)=E(XY)-E(X)E(Y)$

Prof. $Cov(X\,,Y)=E\Big\{(X-E(X))(Y-E(Y))\Big\}$

$\Rightarrow E\{XY-X\,E(Y)-Y\,E(X)+E(X)E(Y)\},$

$\Rightarrow E(XY)-E(X)\,E(Y)-E(Y)\,E(X)+E(X)E(Y),$

$\Rightarrow E(XY)-E(X)\,E(Y),$

So, **Cov(X ,Y) = E(XY)- E(X) E(Y)**

If X and Y are two independent variates the $E(XY)=E(X)\,E(Y)$

$\Rightarrow Cov(X\,,Y)=E(XY)-E(X)E(Y),$

$\Rightarrow Cov(X\,,Y)=E(X)E(Y)-E(X)E(Y)\Rightarrow Cov(X\,,Y)=0.$

So we can say that if X and Y are two independent variates then $Cov(X\,,Y)=0$

Assignment

1. Find the Mean and Variance of the discrete random variable X that has the probability distribution
$P(X=x)=2\,(\frac{1}{3})^x;\quad x=1,2,3,....$

2. Compute the Mean and Variance of the distribution defined by $f(x)=x\,e^{-x};\ x>0.$

4. Find mean and variance for the p.d.f. $f(x)=c\,e^{-x};\ x>0$ where c is an unknown constant. Find c also.

5. Find the Mean and Variance of the discrete random variable X that has the probability distribution
$P(X=x)=\dfrac{\lambda^x\,e^{-\lambda}}{x!};\quad x=0,1,2,3,....$

References

- S.C. Gupta, V.K. Kapoor, Fundamentals of Mathematical Statistics, Sultan Chand & Sons.
- A. Kumar, A. Chaudhary, Probability Distribution & Theory of Attributes, Krishna's Educational Publishers.
- A. Kumar, A. Chaudhary, Probability Distribution & Numerical Analysis, Krishna's Educational Publishers.
- K.S. Negi, Biostatistics, Aitbs Publishers.
- V.B. Rastogi, Fundamentals of Biostatistics, ANE Books.