M.S c Mathematics –SEM 3 Functional Analysis- CC-11 Unit 3

E-content 3–Dr Abhik Singh,

Guest faculty, PG Department of Mathematics, Patna University,

Patna.

Projection on Banach spaces

A projection on a Banach space B is an idempotent operator E on B

In other words E is a projection on B if

(i) $E^2 = E$ i.e E is a projection on B in the algebraic sense

(ii) E is continuous.

Theorem

Let P be a projection on a Banach space B and let M and n be its range and null space respectively.Then M and N are closed linear manifolds (subspace) of B such that B= M⊕N.

Proof

By definition

P is a projection on a Banach space

(i) P is a projection in the algebraic sense and

(ii) P is continuous

But (i) implies that B=M⊕N

where M and N are range and null spaces of P respectively.

Now we use to prove that M and N are closed subspaces of B.

By definition of a null space

$$N = \{x: P(x) = 0\} = P^{-1}(\{0\})$$

Since P is continuous and {0} is closed in B ,it follows that $P^{-1}(\{0\})$ is a closed subspace of B.

We know that in a metric space ,every singleton set is closed.

$$\boldsymbol{M = \{x : P(x) = x\} = \{x : (I - P)(x) = 0}$$

i.e M is the null space of I-P .

But since the identity map I is always continuous

I-P is also continuous. Hence so its null space M must be closed.