e-content (lecture-19)

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MATH SEM-3 CC-11 UNIT-5 (Functional Analysis)

Topic: Normal operators .

Normal operator : An operator T on a Hilbert space H is said to be normal if it commutes with its adjoint

i.e $TT^* = T^*T$.

Now if T is self-adjoint operator then e $T = T^*$

So $TT^* = T^*T$ hence T is normal.

Thus every self-adjoint operator is normal.

Theorem: An operator T on a Hilbert space H is normal

$$\Leftrightarrow \|T^*x\| = \|Tx\| \quad \forall\, x \in H \text{ .}$$

Proof: We have

T is normal$\Leftrightarrow$ $TT^* = T^*T$

$\Leftrightarrow TT^* - T^*T = O$

$\Leftrightarrow ((TT^* - T^*T)x, x) = 0 \;\; \forall\, x \in H$

$\Leftrightarrow (TT^*x - T^*Tx, x) = 0 \;\; \forall\, x \in H$

$\Leftrightarrow (TT^*x, x) - (T^*Tx, x) = 0 \;\; \forall\, x \in H$

$\Leftrightarrow (TT^*x, x) = (T^*Tx, x) \;\; \forall\, x \in H$

$\Leftrightarrow (T^*x, T^*x) = (Tx, Tx) \;\; \forall\, x \in H$

$\Leftrightarrow \|T^*x\| = \|Tx\| \quad \forall\, x \in H.$

Theorem: An operator T on a Hilbert space H can be uniquely expressed as $T = T_1 + iT_2$ where $T_1 and\, T_2$ are self-adjoint operators on H.

Proof : Let $T_1 = \frac{T+T^*}{2}$ $and\, T_2 = \frac{T-T^*}{2i}$

Then $T = T_1 + iT_2$.

Now $T_1^* = [\frac{T+T^*}{2}]^* = \frac{1}{2}[T + T^*]^*$

$$= \frac{1}{2}[T^* + T^{**}]$$

$$= \frac{1}{2}[T^* + T] = T_1.$$

Hence T_1 is self-adjoint operators on H.

Now $T_2^* = [\frac{T-T^*}{2i}]^* = -\frac{1}{2i}[T - T^*]^*$

$$= -\frac{1}{2i}[T^* - T^{**}]$$

$$= -\frac{1}{2i}[T^* - T]$$

$$= \frac{1}{2i}[\, T - T^*] = T_2.$$

Hence T_2 is self-adjoint operators on H.

We have to show that the repressentation is unique .

Let $T = U_1 + iU_2$ be another representation of T.

where $U_1 and\ U_2$ are self-adjoint operators on H.

We have

$$T^* = [U_1 + iU_2\,]^* = U_1^* - iU_2^* = U_1 - iU_2$$

So $T + T^* = (U_1 + iU_2\,) + (U_1 - iU_2\,) = 2U_1$

$$U_1 = \frac{T+T^*}{2} = T_1.$$

Also $T - T^* = (U_1 + iU_2\,) - (U_1 - iU_2\,) = 2iU_2$

$U_2 = \frac{T-T^*}{2i} = T_2$.Hence representation of T is unique.

END.