Complex Potential for Irrotational Motion in Two-dimensions(13)

Binod Kumar[*]
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1 Irrotational Motion in Two-dimensions

Let two-dimensional motion of fluid of velocity $\boldsymbol{q} = (u, v)$ be irrotational, then there exists velocity ϕ such that

$$u = -\frac{\partial \phi}{\partial x} \quad \text{and} \quad v = -\frac{\partial \phi}{\partial y} \tag{1}$$

Here, motion is two dimensional flow , so $\exists$ a stream function ψ such that

$$u = -\frac{\partial \psi}{\partial y} \quad \text{and} \quad v = \frac{\partial \psi}{\partial x} \tag{2}$$

From (1) and (2), we have

$$\frac{\partial \phi}{\partial x} = \frac{\partial \psi}{\partial y} \quad \text{and} \quad \frac{\partial \phi}{\partial y} = -\frac{\partial \psi}{\partial x} \tag{3}$$

Equation (3) represent the Cauchy-Riemann equations for the function $w = f(z) = \phi(x, y) + \iota\psi(x, y)$, where $x, y, \phi, \psi \in \mathbb{R}$ then $w = \phi + \iota\psi$ is analytic.

Remark 1. *Equation (3) can be written as*

$$\frac{\partial \phi}{\partial x}\frac{\partial \psi}{\partial x} + \frac{\partial \phi}{\partial y}\frac{\partial \psi}{\partial y} = 0 \tag{4}$$

showing that the families of curves given by ϕ = constant and ψ = constant intersect orthogonally.
i.e.,

$$d\phi = \frac{\partial \phi}{\partial x}dx + \frac{\partial \phi}{\partial y}dy = 0 \quad \Longrightarrow \quad \left(\frac{dy}{dx}\right)_\phi = -\frac{\frac{\partial \phi}{\partial x}}{\frac{\partial \phi}{\partial y}}$$

[*]Corresponding author, e-mail:binodkumararyan@gmail.com, Telephone: +91-9304524851

and

$$d\psi = \frac{\partial \psi}{\partial x}dx + \frac{\partial \psi}{\partial y}dy = 0 \quad \Longrightarrow \quad \left(\frac{dy}{dx}\right)_\psi = -\frac{\frac{\partial \psi}{\partial x}}{\frac{\partial \psi}{\partial y}}$$

For orthogonal

$$\left(\frac{dy}{dx}\right)_\phi \left(\frac{dy}{dx}\right)_\psi = -1$$

which is represent equation (4)

Remark 2. *The potential and stream function satisfied Laplace equation. Differentiating equation (3), we get*

$$\frac{\partial^2 \phi}{\partial x^2} = \frac{\partial^2 \psi}{\partial x \partial y} \quad \text{and} \quad \frac{\partial^2 \phi}{\partial y^2} = -\frac{\partial^2 \psi}{\partial y \partial x} \tag{5}$$

Since $u = -\frac{\partial \psi}{\partial y}$ *and* $v = \frac{\partial \psi}{\partial x}$. *Then*

$$\frac{\partial}{\partial x}u + \frac{\partial}{\partial y}v = 0 \quad \Longrightarrow \quad \frac{\partial^2 \psi}{\partial x \partial y} = \frac{\partial^2 \psi}{\partial y \partial x}$$

Then equation (5) gives

$$\frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial y^2} = \frac{\partial^2 \psi}{\partial x \partial y} + \frac{\partial^2 \psi}{\partial y \partial x} = 0$$
$$\nabla^2 \phi = \frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial y^2} = 0$$

Similarly for function ψ, *we get*

$$\nabla^2 \psi = \frac{\partial^2 \psi}{\partial x^2} + \frac{\partial^2 \psi}{\partial y^2} = 0$$

1.1 Complex Potential

Let $w = f(z) = \phi + \iota\psi$ be taken as a function of $x + \iota y\ (= z)$ is said to be complex potential. Which is analytic for this

$$\phi + \iota\psi = f(x + \iota y) \tag{6}$$

Differentiating w.r.t. x and y respectively, we get

$$\frac{\partial \phi}{\partial x} + \iota\frac{\partial \phi}{\partial x} = f'(x + \iota y)$$
$$\frac{\partial \phi}{\partial y} + \iota\frac{\partial \phi}{\partial y} = \iota f'(x + \iota y)$$
$$\Longrightarrow \quad \frac{\partial \phi}{\partial x} = \frac{\partial \psi}{\partial y} \quad \text{and} \quad \frac{\partial \phi}{\partial y} = -\frac{\partial \psi}{\partial x} \tag{7}$$

1.1.1 Cauchy-Riemann equation in polar form

Let (r, θ) be coordinate in polar form such that $r = |z| = \sqrt{x^2 + y^2}$ and $\theta = \tan^{-1}\left(\frac{y}{x}\right)$, then $z = re^{\iota\theta}$, then complex potential

$$\phi(r, \theta) + \iota\psi(r, \theta) = f(re^{\iota\theta}) \tag{8}$$

Differentiating w.r.t. r and θ respectively, we get

$$\frac{\partial \phi}{\partial r} + \iota\frac{\partial \psi}{\partial r} = e^{\iota\theta} f'(re^{\iota\theta})$$

$$\frac{\partial \phi}{\partial \theta} + \iota\frac{\partial \psi}{\partial \theta} = \iota re^{\iota\theta} f'(re^{\iota\theta})$$

$$\implies \quad \frac{\partial \phi}{\partial \theta} + \iota\frac{\partial \psi}{\partial \theta} = \iota r\left(\frac{\partial \phi}{\partial r} + \iota\frac{\partial \psi}{\partial r}\right)$$

$$\therefore \quad \frac{\partial \phi}{\partial r} = \frac{1}{r}\frac{\partial \psi}{\partial \theta} \quad \text{and} \quad \frac{\partial \phi}{\partial \theta} = r\frac{\partial \psi}{\partial r} \tag{9}$$

is Cauchy-Riemann (C-R) equation in polar form.

1.1.2 Magnitude of velocity

Let $w = f(z)$ be the complex potential. Then

$$w = \phi + \iota\psi \quad \text{and} \quad z = x + \iota y \tag{10}$$

Also

$$\partial\phi/\partial x = \partial\psi/\partial y \quad \text{and} \quad \partial\phi/\partial y = -\partial\psi/\partial x \tag{11}$$

For two- dimensional irrotational motion, we have

$$u = -\partial\phi/\partial x \quad \text{and} \quad v = -\partial\phi/\partial y \tag{12}$$

From equation (10)

$$\frac{dw}{dz}\frac{\partial z}{\partial x} = \frac{\partial \phi}{\partial x} + \iota\frac{\partial \psi}{\partial x} \quad \text{and} \quad \frac{\partial z}{\partial x} = 1$$

Using equation (11)

$$\frac{dw}{dz} = \frac{\partial \phi}{\partial x} - \iota\frac{\partial \phi}{\partial x}$$

$$\frac{dw}{dz} = -u + \iota v$$

which is called the complex velocity. And magnitude of complex velocity is

$$\left|\frac{dw}{dz}\right| = \sqrt{\frac{\partial \phi}{\partial x}^2 + \frac{\partial \phi}{\partial x}^2} = \left(u^2 + v^2\right)^{1/2} = q$$

All the best...

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