e-content (lecture-18)

by

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MATH SEM-3 CC-11 UNIT-5 (Functional Analysis)

Topic: Positive Operators

Definition: A self-adjoint operator T on a Hilbert space H is said to be positive if $(Tx, x) \geq 0 \quad \forall \; x \in H$.

Example: Identity (I) and zero(O) operators are both positive operator .

we know that $I \; and \; O$ are self-adjoint operators.

Also we have $\forall \; x \in H$

$$(Ix, x) = (x, x) = \|x\|^2 \geq 0 \quad \text{and}$$

$$(Ox, x) = (0, x) = 0.$$

Hence $I \; and \; O$ are positive operators.

Theorem: If T is a operator positive on a Hilbert space H,then $I+T$ is non-singular.

Proof : To prove $I+T$ is non-singular we have to show that $I+T$ is one- one onto as a mapping of H to itself.

$I+T$ is one- one: Let $x, y \in H$ such that

$$(I+T)x = (I+T)y$$

$$\Rightarrow (I+T)x - (I+T)y = 0$$

$$\Rightarrow (I+T)(x-y) = 0$$

$$\Rightarrow (I+T)(\alpha) = 0 \quad \text{[where } \alpha = x - y\text{]}$$

$$\Rightarrow I\alpha + T\alpha = 0$$

$$\Rightarrow T\alpha = -\alpha$$

So $(T\alpha, \alpha) = (-\alpha, \alpha) = -\|\alpha\|^2$

Now T is a operator positive

So $\quad (T\alpha, \alpha) \geq 0$

$$\Rightarrow -\|\alpha\|^2 \geq 0$$

$$\Rightarrow \|\alpha\|^2 \leq 0$$

$$\Rightarrow \|\alpha\|^2 = 0$$

$$\Rightarrow \alpha = 0 \ \Rightarrow x - y = 0 \Rightarrow x = y.$$

Hence $I + T$ is one- one.

To prove $I + T$ is onto.

Let M be the range of $I + T$.

We show that $M = H$. First we show that M is closed.

For any vector $x \in H$ we have

$$\|(I + T)x\|^2 = \|(Ix + Tx)\|^2$$

$$= \|x + Tx\|^2$$

$$= (x + Tx, x + Tx)$$

$$= (x, x) + (Tx, x) + (x, Tx) + (Tx, Tx)$$

$$= \|x\|^2 + \|Tx\|^2 + (Tx, x) + \overline{(Tx, x)}$$

$$= \|x\|^2 + \|Tx\|^2 + 2(Tx, x) \text{ [as } (Tx, x) \text{ is real]}$$

$$\geq \|x\|^2$$

Thus $\|x\|^2 \leq \|(I + T)x\|^2$

Hence $\|x\| \leq \|(I + T)x\| \quad \forall \; x \in H$.

Now let $((I + T)\, x_n)$ be a chauchy sequence in M.

Then we have for all $m, n \in N$

$$\|x_m - x_n\| \leq \|(I + T)(x_m - x_n)\|$$

$$\leq \|(I+T)(x_m) - (I+T)x_n)\| \to 0$$

$\Rightarrow \|x_m - x_n\| \to 0$ so $(\, x_n)$ be a chauchy sequence in H

Hhence there exists $x \in H$ such that $x_n \to x$.

$$\lim_{n\to\infty} [(I+T)\, x_n] = (I+T)\left(\lim_{n\to\infty} x_n\right)$$

$$= (I+T)x \in M$$

So M is complete therefore it is closed .

Now suppose that $M \neq H$ then M is a proper closed subspace of H hence there exists a non zero vector

$x_0\; in\; H$ such that x_0 is orthogonal to M.

Since $(I+T)x_0 \;\in M$

Hence $0 = \left((I+T)x_0\,, x_0\right) = (Ix_0\; + T\, x_0\,, x_0\,)$

$$= (x_0\; + T\, x_0\,, x_0\,)$$

$$= (x_0\,, x_0\,) + (T\, x_0\,, x_0\,)$$

$$\Rightarrow -\|x_0\,\|^2 = (T\, x_0\,, x_0\,) \geq 0$$

$$\Rightarrow \|x_0\,\|^2 \leq 0$$

$$\Rightarrow \|x_0\,\|^2 = 0$$

$$\Rightarrow \|x_0\,\| = 0$$

$$\Rightarrow x_0 = 0$$

This is a contradiction hence we must have $M = H$.

Therefore $I + T$ is onto.

END.