PDE : (M.Sc. Sem-IV)

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GENERAL NATURE OF BOUNDARY VALUE PROBLEMS

The three popular types of BC'S in Boundary = Value problems

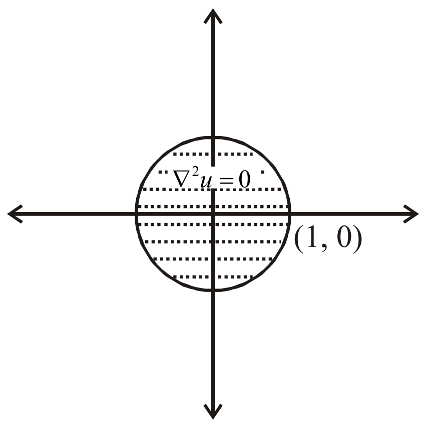
(i) Boundary value problems of first kind (Dirichlet Problem) :–

In Dirichlet's type problem; the PDE is defined over a given region of space and solution is specified on the boundary of the region.

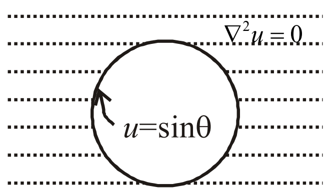
Example (1) : (Interior Dirichlet's problem)

PDE : $u_{rr} + \frac{1}{r}u_r + \frac{1}{r^2}u_{\theta\,\theta} = 0; \quad 0 < r < 1$

BC'S : $u(1, \theta) = \sin\theta; \; 0 \le \theta \le 2\pi$



Example (2) :– (Exterior Dirichlet problem)



Here we have to solve Laplace's equation $\nabla^2 u = 0$ (outside the) unit circle & $u(1, \theta) = \sin\theta$

(ii) Second kind (Neumann Problem)

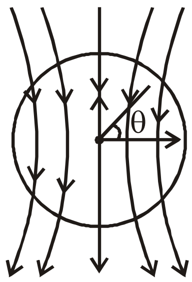
In Neumann types of problem the PDE is defined over some region of given space and this time at the boundary of region the outward normal derivative $\frac{\partial u}{\partial n}$ is specified .

Example (3) :– The steady state temperature inside the circle would then be given by the solution of BVP.

$$\nabla^2 u = 0;\ 0 < r < 1$$

$$\frac{\partial u}{\partial r} = \sin\theta;\ r = 1\ \&\ 0 \le \theta \le 2\pi$$

Note : The outward normal derivative $\frac{\partial u}{\partial n}$ thus proportional to the inward flux



Here we observe that the flux of heat (cal/cm sec) across the boundary is inward for $0 \le \theta < \pi$ & outward for $\pi \le \theta < 2\pi$

Also; since total flux : $\int_0^{2\pi} \frac{\partial u}{\partial r} d\theta = \int_0^{2\pi} \sin\theta d\theta = 0$

Note :– Neumann problems make sense only if the net gain in heat across the boundary is zero.

i.e., $\int_c \frac{\partial u}{\partial n} = 0$ (must be true) .... e_1

Example (4) :

$$\nabla^2 u = 0;\ 0 < r < 1$$

$$\frac{\partial u}{\partial r}(1,\ \theta) = 1;\ r = 1 \text{ and } 0 \le \theta \le 2\pi$$

has no physical meaning by (e_1)

(iii) Boundary value problems of the third kind :–

In this types of problem, the PDE is defined over some region of space and the boundary condition is a mixture of Dirichlet & Neumann move over the boundary condition can be stated as $\frac{\partial u}{\partial n} + h(u - g) = 0$ where h is a constant and g a given function that can vary over the boundary. A more suggestive form of this BC'S would be

$$\frac{\partial u}{\partial n} + h(u - g) = 0$$

Which reveals the inward flux across the boundary is proportional to the difference between the temprature u and some specified temperature g this also explain that

(i) If the temperature u is greater than the boundary temperature, then the flow of heat is outward.

(ii) If u is less than the boundary temperature g then flows inward.

Assignment (i) : Solve the Dirichlet problem

$$\text{PDE} \quad \nabla^2 u = 0 \;\; ; \;\; 0 < r < 1$$

$$\text{BC's} \quad u(1, \theta) = \sin\theta \;\; ; \;\; 0 \le \theta < 2\pi$$

Assignment (ii) : For different value of h. Describe the solution of $u(r, \theta)$ for

$$\text{PDE} \quad \nabla^2 u = 0 \;\; ; \;\; 0 < r < 1$$

$$\frac{\partial u}{\partial r} = \sin^2\theta \;\; ; \;\; 0 \le \theta < 2\pi$$