e-content (lecture-17)

by

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MATH SEM-3 CC-11 UNIT-4 (Functional Analysis)

Topic:Theorems on operator and self- adjoint operators.

Theorem: If T is an operator on a Hilbert space H,

then $T = O \Leftrightarrow (Tx, y) = 0 \;\forall\; x, y \in H$.

Proof: Suppose that $T = O$ $then\ for\ all\ x, y \in H$.

We have $(Tx, y) = (Ox, y) = (0, y) = 0$.

Conversely,

Suppose that $(Tx, y) = 0 \;\forall\; x, y \in H$.

Then $(Tx, Tx) = 0 \;\forall\; x, \in H$. [taking $y = Tx$]

$\Rightarrow Tx = 0 \quad \forall\; x, \in H$.

$\Rightarrow T = O$.

Theorem: If T is an operator on a Hilbert space H,

then $T = O \Leftrightarrow (Tx, x) = 0 \;\forall\; x \in H$.

Proof: Suppose that $T = O$ $then\ for\ all\ x \in H$.

We have $(Tx, x) = (Ox, x) = (0, x) = 0$.

Conversely,

Suppose that $(Tx, x) = 0 \;\forall\; x \in H$.

We have for all scalars $\alpha\ and\ \beta$, and $\forall\; x, y \in H$

$$0 = (T(\alpha x + \beta y), \alpha x + \beta y)$$

$$= (\alpha Tx + \beta Ty, \alpha x + \beta y)$$

$$= \alpha(Tx, \alpha x + \beta y) + \beta(Ty, \alpha x + \beta y)$$

$$= \alpha\overline{\alpha}(Tx, x) + \alpha\overline{\beta}(Tx, y) + \beta\overline{\alpha}(Ty, x) + \beta\,\overline{\beta}(Ty, y)$$

$$= \alpha\overline{\beta}(Tx, y) + \beta\overline{\alpha}(Ty, x) \text{ [since } (Tx, x) = 0 \;\forall\; x \in H]$$

$$\alpha\overline{\beta}(Tx, y) + \beta\overline{\alpha}(Ty, x) = 0 \ldots \ldots ..(1)$$

This is true for all scalars $\alpha\ and\ \beta$, and $\forall\; x, y \in H$.

So putting $\alpha = 1\ and\ \beta = 1\ \ in\ (1)$ we get

$$(Tx, y) + (Ty, x) = 0 \ldots \ldots ..(2)$$

Again putting $\alpha = i\ and\ \beta = 1\ \ in\ (1)$ we get

$$i(Tx, y) - i(Ty, x) = 0 \ldots \ldots . . (3)$$

multiplying (2) by i and adding (3) we get

$$2i(Tx, y) = 0 \quad \forall \ x, y \in H$$

$$\Rightarrow (Tx, y) = 0 \quad \forall \ x, y \in H$$

$$\Rightarrow (Tx, Tx) = 0 \quad \forall \ x \in H \quad taking\ y = Tx$$

$$\Rightarrow (Tx, x) = 0 \quad \forall \ x \in H$$

$$\Rightarrow T = O.$$

Theorem: If T is an operator on a Hilbert space H,

then T is self- adjoint$\Leftrightarrow$ $(Tx, x) is\ real \ \forall \ x \in H$.

Proof : suppose that T is self- adjoint operator on H.

So $T^* = T$

Then $\forall \ x \in H$ we have

$$(Tx, x) = (x, T^*x) = (x, Tx) = \overline{(x, Tx)}$$

Hence $(Tx, x) is\ real \ \forall \ x \in H$.

Conversely suppose that $(Tx, x) is\ real \ \forall \ x \in H$.

$$(Tx, x) = \overline{(Tx, x)}$$

$$= \overline{(x, T^*x)} = (T^*x, x) \ \forall \ x \in H.$$

$$\Rightarrow (Tx, x) - (T^*x, x) = 0 \quad \forall \ x \in H$$

$$\Rightarrow (Tx - T^*x, x) = 0 \quad \forall \ x \in H$$

$$\Rightarrow ((T - T^*)x, x) = 0 \quad \forall \ x \in H$$

$$\Rightarrow T - T^* = 0 \ \Rightarrow T = T^*$$

T is self- adjoint operator on H.

END.