e-content (lecture-16)

by

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MATH SEM-3 CC-11 UNIT-4 (Functional Analysis)

Topic: Theorems on self –adjoint operators.

Theorem: The set S of all self adjoint operators in B(H) is a closed real linear subspace of B(H) and therefore a real Banach space which contains the identity transformation .

Proof: we know that O (zero operator) is self adjoint operator so O belongs to S hence S is non-empty set

Let A_1 and A_2 be any two self adjoint operators on H.

So $A_1^* = A_1$ and $A_2^* = A_2$.

Let α and β be any two real scalars .

then $\overline{\alpha} = \alpha$ and $\overline{\beta} = \beta$

$$(\alpha A_1 + \beta A_2)^* = \overline{\alpha}\, A_1^* + \overline{\beta} A_2^* = \alpha A_1 + \beta A_2.$$

So $\alpha A_1 + \beta A_2$ is self adjoint operators on H.

Hence $\alpha A_1 + \beta A_2$ belongs to S

Therefore S is a real linear subspace of B(H).

Now we show that S is closed .

Let A be any limit point of S. Then there exists a sequence (A_n) in S such that $A_n \to A$.

We have

$$\begin{aligned}
\|A - A^*\| &= \|(A - A_n) + (A_n - A^*)\| \\
&\leq \|(A - A_n)\| + \|(A_n - A^*)\| \\
&= \|(A - A_n)\| + \|(A_n - A_n^*) + (A_n^* - A^*)\| \\
&\leq \|(A - A_n)\| + \|(A_n - A_n^*)\| + \|(A_n^* - A^*)\| \\
&= \|(A - A_n)\| + \|O\| + \|(A - A_n)^*\| \\
&= \|(A - A_n)\| + 0 + \|A - A_n\| \\
&= 2\|(A - A_n)\| \to 0 \text{ as } A_n \to A \, .
\end{aligned}$$

So $\|A - A^*\| = 0 \Rightarrow A - A^* = 0 \Rightarrow A = A^*.$

Hence A is self adjoint operators on H.

so A belongs to S therefore S is closed .

since S is a closed real linear subspace of of a complete space B(H) therefore S is also complete space . Hence

S is a real banach space Also $I = I^*$ so I belongs to S .

Theorem: Let A_1 *and* A_2 be any two self adjoint operators on H then their product

$$A_1\, A_2 \text{ is self adjoint} \Leftrightarrow A_1\, A_2 = A_2\, A_1$$

i.e A_1 *and* A_2 commute.

Proof: Let A_1 *and* A_2 be any two self adjoint operators on H then $A_1^* = A_1$ *and* $A_2^* = A_2$.

Suppose that A_1 *and* A_2 commute.

Then $(A_1\, A_2)^* = A_2^* A_1^* = A_2\, A_1 = A_1\, A_2$.

So $A_1\, A_2$ is self adjoint.

Conversely, Suppose that $A_1\, A_2$ is self adjoint.

To prove that A_1 *and* A_2 commute.

Since $A_1\, A_2$ is self adjoint .

So $A_1\, A_2 = (A_1\, A_2)^* = A_2^* A_1^* = A_2\, A_1$.

Hence A_1 *and* A_2 commute. **END**