M.S c Mathematics –SEM 3 Functional Analysis-CC-11 Unit 2

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Example of Conjugate Space

Prove that the dual space of c_0 is l_1

Proof

let $f \in c_0^*$(the dual space of C_0) .It is sufficient to show that there exist a=$(a_n) \in l_1$such that

$f(x) = \sum_{n=1}^{\infty} x_n a_n$ for all and $||f|| = ||a||$

Let $\delta_m^n = \{$1 if n=m $x = (x_n) \in C_0$

0 if n$\neq m$

And let $\partial^k = (0, 0 \ldots . 0, 1, 0, 0 \ldots . .)$where the number 1 appears as the k th co-0rdinates and all other c0-ordinates are 0.

Put $a_n = f(\delta^n)$

Let $x = (x_n) \in C_0$ and let

$$y_n = \sum_{k=1}^{n} x_k^{(k)}$$

Then $f(y_k) = f(\sum_{k=1}^{n} \frac{x_k^{(k)} =}{\sum_{k=1}^{n} x_k} f(\partial^{(k)})$

$= \sum_{k=1}^{n} x_k a_k.$

Now $x - y_n = (0, 0 \dots 0\ , x_{n+1,} x_{n+2} \dots .)$ and thus $||x - y_n|| = \lim_{n\to\infty} lub\{|x_k|: k > n\} = \lim_{n\to\infty} sup|x_n| = 0$

Since $(x_n) \in C_0$

Thus $\lim_{n\to\infty} y_n = x$ in c_0 . Since f is continuous.

f(x) = $\lim_{n\to\infty} f(y_n) =$
$\lim_{n\to\infty} \sum_{k=1}^{n} x_k\, a_k$=$\sum_{n=1}^{\infty} x_n a_n$

Let $\gamma_m^{(n)} =$ 1 if m$\leq$ n and a_n $\geq$0

-1 if m$\leq$ n and a_n $<$0

0 if m>n

Then $f(\gamma^{(n)}) = \sum_{m=1}^{\infty} \gamma_m^{(n)} a_m = \sum_{m=1}^{n} |a_m|$

Now $\gamma^n \epsilon C_0$ **and** $||\gamma^n|| = 1$

Hence $|f(\gamma^n)| \leq ||f||$

Thus $\sum_{m=1}^{n} |a_m| \leq ||f||$ **for n=1,2,3..........(i)**

Hence a=$(a_n) \in l_1$

Taking limit

$||a|| = \sum_{n=1}^{\infty} |a_n| \leq ||f||$**..........(ii)**

If x= $(x_n) \in C_0$ **and** $||x|| = 1$

| f(x) |=$|\sum_{n=1}^{\infty} x_n a_n| \leq \sum_{n=1}^{\infty} |x_n||a_n|$

$\leq \sum_{n=1}^{\infty} |a_n| = ||a||$

Hence $||f|| \leq ||a||$ **........(iii)**

From (ii) and (iii) we have ||f||=||a||

Therefore the dual space of C_0 **may be identified with** l_1 **.**