Motion in Two-Dimensions Source and Sink(12)

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1 Stream function or current function

Let u and v be the components of velocity in two-dimensional motion. The the differential equation of streamline flow, is given by

$$\frac{dx}{u} = \frac{dy}{v} \quad \text{or} \quad vdx - udy = 0 \tag{1}$$

and the equation of continuity of fluid flow is

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0 \quad \text{or} \quad \frac{\partial v}{\partial y} = \frac{\partial(-u)}{\partial x} \tag{2}$$

Equation (2) be exact condition for the differential equation (1). so solution of equation must be in the form $d\psi = 0$ such that

$$d\psi = \frac{\partial \psi}{\partial x}dx + \frac{\partial \psi}{\partial y} = vdx - udy = 0 \tag{3}$$

so that

$$u = -\frac{\partial \psi}{\partial y} \qquad v = \frac{\partial \psi}{\partial x}$$

and

$$d\psi = 0 \qquad \Longrightarrow \psi = C \quad \text{(Constant)}$$

This function ψ is known as stream function. Thus the stream function is constant along a streamline motion. Clearly current function is exist by virtue of the equation of continuity and incompressibile=ity of the fluid. Hence current function exists in all type of two-dimensional motion wheather rotational or irrotational.

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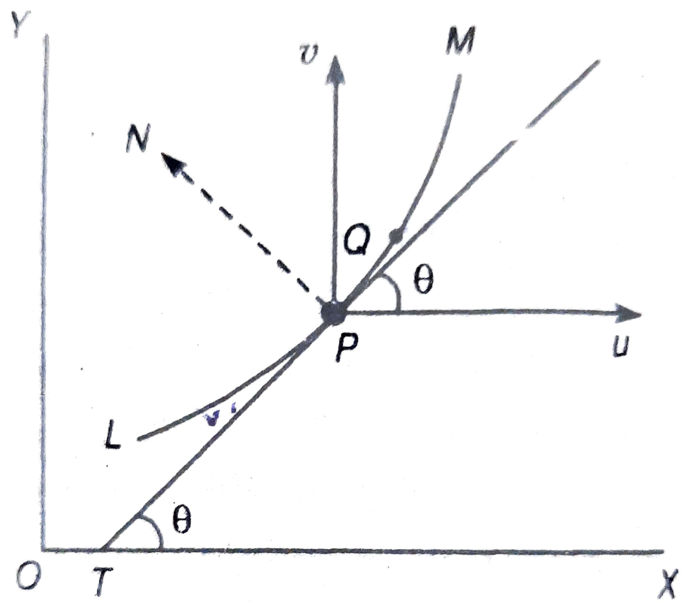
1.1 Physical Significance of stream function

Let LM be any curve in the $x-y$ plane and ψ_1 and ψ_2 be stream function at L and M respectively. Let P be an arbitrary point on LM such that arc$LP = s$ and Q be a neighbouring point on LM such that arc$LQ = s+\delta s$. Let θ be the angle between tangent at P and $x-$ axis. If u and v be the velocity components at point P, then



$$\text{velocity at P along inward drawn normal PN} = v\cos\theta - u\sin\theta \tag{4}$$

when ψ is the stream function, then we have

$$u = -\frac{\partial\psi}{\partial y} \quad \text{and} \quad v = \frac{\partial\psi}{\partial x} \tag{5}$$

Also from calculus

$$\cos\theta = \frac{dx}{ds} \quad \text{and} \quad \sin\theta = \frac{dy}{ds} \tag{6}$$

Using equation (4), we get

$$\text{flux across PQ from right to left} = (v\cos\theta - u\sin\theta)\delta s \tag{7}$$

$\therefore$ Total flux for curve LM from right to left

$$= \int_{LM}(v\cos\theta - u\sin\theta)ds = \int_{LM}\Big(\frac{\partial\psi}{\partial x}\frac{dx}{ds} + \frac{\partial\psi}{\partial y}\frac{dy}{ds}\Big)ds$$

$$\int_{LM} \Big(\frac{\partial\psi}{\partial x}dx + \frac{\partial\psi}{\partial y}ds = \int_{\psi_1}^{\psi_2} \Big(\frac{\partial\psi}{\partial x}dx + \frac{\partial\psi}{\partial y}ds\Big) = \int_{\psi_1}^{\psi_2} d\psi = \psi_1 - \psi_2$$

Thus a property of the current function is that the difference of its values at two points respond the flow across any line joining the points.

Remark 1. *Since the velocity normal to δs will contribute to the flux across δs where as the velocity along tangent to δs will not contribute towards flux across δs, we have*

$$\begin{aligned} \textit{flux across} \;\; \delta s &= \delta s \times \textit{normal velocity} \\ (\psi + \delta\psi) - \psi &= \delta s \times \;\textit{velocity from right to left across} \\ \textit{velocity from right to left across} \;\; \delta s &= \frac{\partial\psi}{\partial s} \end{aligned} \tag{8}$$

Remark 2. *Velocity components in terms of ψ in plane-polar coordinate (r, θ) can be obtained by using the method in remarks (1). Let q_r and q_θ be velocity components in the direction r and θ increasing respectively. Then*

$$\begin{aligned} q_r &= \;\textit{velocity from right to left across} \;\; r\delta\theta \\ &= \lim_{\delta\theta\to 0} \frac{\delta\psi}{r\delta\theta} = \frac{1}{r}\frac{\partial\psi}{\partial\theta} \\ \textit{and} \qquad q_\theta &= \;\textit{velocity from right to left across} \;\; \delta r \\ &= \lim_{\delta r\to 0} \frac{\delta\psi}{\delta r} = \frac{\partial\psi}{\partial r} \\ \textit{Thus} \qquad q_r = \frac{1}{r}\frac{\partial\psi}{\partial\theta} \qquad \textit{and} \qquad q_\theta &= \frac{\partial\psi}{\partial r} \end{aligned} \tag{9}$$

1.2 Basic concept of complex valued function of complex variable

Suppose $z = x + \imath y$ and that $w = f(z) = \phi(x, y) + \imath\psi(x, y)$, where $x, y, \phi, \psi \in \mathbb{R}$. Also suppose that ϕ and ψ and their derivatives are everywhere continuous within a domain. If at any point Z of its domain the derivatives $dw/dz = f'(z)$ is unique, then w is is said to be analytic at that point. If w is analytic throughout the domain is said analytic or regular throughout the domain. If this is analytic throughout the domain then must satisfied the necessary and sufficient condition for w

$$\frac{\partial\phi}{\partial x} = \frac{\partial\psi}{\partial y} \;\; \text{and} \;\; \frac{\partial\phi}{\partial y} = -\frac{\partial\psi}{\partial x} \tag{10}$$

which are known Cuchy-Riemann (C-R) equations. The function ϕ and ψ are known as conjugate functions.

Remark 3.

$$f_x(z) = \frac{\partial \phi}{\partial x} + \iota \frac{\partial \psi}{\partial x}$$

and

$$f_y(z) = \frac{\partial \phi}{\partial y} + \iota \frac{\partial \psi}{\partial y}$$

$$\implies \quad \frac{\partial \phi}{\partial y} + \iota \frac{\partial \psi}{\partial y} = \iota(\frac{\partial \phi}{\partial x} + \iota \frac{\partial \psi}{\partial x})$$

We get another form of C-R equation

$$\iota f_x(z) = f_y(z)$$

Remark 4. *Since* $z = x + \iota y$ *that implies* $\bar{z} == x - \iota y$, *then* $x = \frac{z+\bar{z}}{2}$ *and* $y = \frac{z-\bar{z}}{2\iota}$ *We get We get another form of C-R equation*

$$\frac{\partial^2 f}{\partial z \partial \bar{z}} = 0$$

All the best...
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