Bernoulli's equation and its application (11)

Binod Kumar[*]
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1 Bernoulli's equation and its application

1.1 Integration of Euler's equation of motion

When a velocity potential (ϕ) exists(i.e., motion is irrotational) and the external forces ($\boldsymbol{F} = (X, Y, Z)$) are derivable from potential function (V), the equation of motion can always be integrated. Let $\boldsymbol{q} = (u, v, w)$ be velocity, Then by definition, we get

$$u = \frac{\partial \phi}{\partial x}, \qquad v = \frac{\partial \phi}{\partial y}, \qquad w = \frac{\partial \phi}{\partial z} \tag{1}$$

$$X = -\frac{\partial V}{\partial x}, \qquad Y = -\frac{\partial V}{\partial y}, \qquad Z = -\frac{\partial V}{\partial z} \tag{2}$$

and

$$\frac{\partial u}{\partial y} = \frac{\partial v}{\partial x}, \qquad \frac{\partial v}{\partial z} = \frac{\partial w}{\partial y}, \qquad \frac{\partial w}{\partial x} = \frac{\partial u}{\partial z} \tag{3}$$

Then by Euler's dynamical equations are

$$\begin{aligned}
\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} + w\frac{\partial u}{\partial z} &= X - \frac{1}{\rho}\frac{\partial p}{\partial x} \\
\frac{\partial v}{\partial t} + u\frac{\partial v}{\partial x} + v\frac{\partial v}{\partial y} + w\frac{\partial v}{\partial z} &= Y - \frac{1}{\rho}\frac{\partial p}{\partial y} \\
\frac{\partial w}{\partial t} + u\frac{\partial w}{\partial x} + v\frac{\partial w}{\partial y} + w\frac{\partial w}{\partial z} &= Z - \frac{1}{\rho}\frac{\partial p}{\partial z}
\end{aligned}$$

[*]Corresponding author, e-mail:binodkumararyan@gmail.com, Telephone: +91-9304524851

Using equations (1),(2) and (3)

$$\left.\begin{array}{l} -\frac{\partial^2 u}{\partial t \partial} + u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial x} + w\frac{\partial u}{\partial x} = -\frac{\partial V}{\partial x} - \frac{1}{\rho}\frac{\partial p}{\partial x} \\ -\frac{\partial^2 v}{\partial t \partial y} + u\frac{\partial v}{\partial y} + v\frac{\partial v}{\partial y} + w\frac{\partial v}{\partial y} = -\frac{\partial V}{\partial y} - \frac{1}{\rho}\frac{\partial p}{\partial y} \\ -\frac{\partial^2 w}{\partial t \partial z} + u\frac{\partial w}{\partial z} + v\frac{\partial w}{\partial z} + w\frac{\partial w}{\partial z} = -\frac{\partial V}{\partial z} - \frac{1}{\rho}\frac{\partial p}{\partial z} \end{array}\right\} \tag{4}$$

Using

$$\frac{1}{2}\frac{\partial}{\partial x}\left(u^2+v^2+w^2\right) = u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial x} + w\frac{\partial u}{\partial x}$$

$$\frac{1}{2}\frac{\partial}{\partial y}\left(u^2+v^2+w^2\right) = u\frac{\partial v}{\partial y} + v\frac{\partial v}{\partial y} + w\frac{\partial v}{\partial y}$$

$$\frac{1}{2}\frac{\partial}{\partial z}\left(u^2+v^2+w^2\right) = u\frac{\partial w}{\partial z} + v\frac{\partial w}{\partial z} + w\frac{\partial w}{\partial z}$$

Then equation (4) becomes

$$-\frac{\partial}{\partial x}\left(\frac{\partial \phi}{\partial t}\right) + \frac{1}{2}\frac{\partial}{\partial x}\left(u^2+v^2+w^2\right) = -\frac{\partial V}{\partial x} - \frac{1}{\rho}\frac{\partial p}{\partial x} \tag{5}$$

$$-\frac{\partial}{\partial y}\left(\frac{\partial \phi}{\partial t}\right) + \frac{1}{2}\frac{\partial}{\partial y}\left(u^2+v^2+w^2\right) = -\frac{\partial V}{\partial y} - \frac{1}{\rho}\frac{\partial p}{\partial y} \tag{6}$$

$$-\frac{\partial}{\partial z}\left(\frac{\partial \phi}{\partial t}\right) + \frac{1}{2}\frac{\partial}{\partial z}\left(u^2+v^2+w^2\right) = -\frac{\partial V}{\partial z} - \frac{1}{\rho}\frac{\partial p}{\partial z} \tag{7}$$

Now,

$$d\left(\frac{\partial \phi}{\partial t}\right) = \frac{\partial}{\partial x}\left(\frac{\partial \phi}{\partial t}\right)dx + \frac{\partial}{\partial y}\left(\frac{\partial \phi}{\partial t}\right)dy + \frac{\partial}{\partial z}\left(\frac{\partial \phi}{\partial t}\right)dz \tag{8}$$

$$dV = \frac{\partial v}{\partial x}dx + \frac{\partial v}{\partial y}dy + \frac{\partial v}{\partial z}dz \tag{9}$$

$$dp = \frac{\partial p}{\partial x}dx + \frac{\partial p}{\partial y}dy + \frac{\partial p}{\partial z}dz \tag{10}$$

$$d\left(u^2+v^2+w^2\right) = \frac{\partial}{\partial x}\left(u^2+v^2+w^2\right)dx + \frac{\partial}{\partial y}\left(u^2+v^2+w^2\right)dy + \frac{\partial}{\partial z}\left(u^2+v^2+w^2\right)dz \tag{11}$$

Multiplying equations (5), (6) and (7) by dx, dy and dz respectively, then adding and using equations (8), (9) and (10), we have

$$-d\left(\frac{\partial \phi}{\partial t}\right) + \frac{1}{2}\frac{\partial}{\partial x}\left(u^2+v^2+w^2\right) = -dV - \frac{1}{\rho}dp$$

$$\text{or,} \quad -d\left(\frac{\partial \phi}{\partial t}\right) + \frac{1}{2}dq^2 + dV + \frac{1}{\rho}dp = 0 \tag{12}$$

where $\boldsymbol{q}\cdot\boldsymbol{q} = q^2 = (u^2+v^2+w^2)$ = square of velocity of fluid particle.
If ρ is a function of p. Then integrate equation (12)

$$-\frac{\partial \phi}{\partial t} + \frac{1}{2}q^2 + V + \int \frac{dp}{\rho} = F(t) \tag{13}$$

where $F(t)$ is an arbitrary function of t arising from integration constant. Equation (13) is Bernoulli's equation in most general form.

Case I. Let the fluid be homogeneous and inelastic (so that ρ = Constant i.e., fluid is incompressible). The Bernoulli's equation for unsteady and irrotational motion is given by

$$-\frac{\partial \phi}{\partial t} + \frac{1}{2}q^2 + V + \int \frac{dp}{\rho} = F(t)$$

Case II. If the motion is steady $\frac{\partial \phi}{\partial t} = 0$. The Bernoulli's equation for steady and irrotational motion of an incompressible fluid,is given by

$$\frac{q^2}{2} + V + \frac{p}{\rho} = F(t)$$

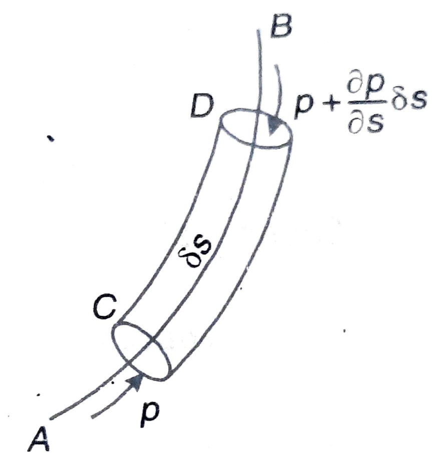
1.2 Bernoulli's Theorem(Steady motion with no velocity potential and conservatives field force)

Statement 1. *When the motion is steady and the velocity potential does not exist, we have*

$$\frac{1}{2}q^2 + V + \int \frac{dp}{\rho} = C$$

where V is the force potential from which the external force are derivable

Proof. Consider a streamline AB in the fluid. Let δs be an element of the stream line and CD be a small cylinder of cross- sectional area α and δs as axis. If $\boldsymbol{q}$ be the velocity and S be the component of external fore per unit mass in the direction of the stremline, then by Newton's second law of motion, we have



$$\rho \delta s \frac{d\boldsymbol{q}}{dt} = \rho \delta s S + p\alpha - \left(p + \frac{\partial p}{\partial s}\delta s\right)\alpha$$

$$\text{or,} \quad \frac{d\boldsymbol{q}}{dt} = S - \frac{1}{\rho}\frac{\partial p}{\partial s}$$

$$\text{or} \quad \frac{\partial \boldsymbol{q}}{\partial t} + \boldsymbol{q}\frac{\partial \boldsymbol{q}}{\partial s} = S - \frac{1}{\rho}\frac{\partial p}{\partial s} \tag{14}$$

If the motion be steady $\frac{\partial \boldsymbol{q}}{\partial t} = 0$ and if the external force have a potential function V such that $S = -\frac{\partial V}{\partial s}$, then equation (14) reduced to

$$\frac{\partial \boldsymbol{q}^2}{\partial s} + \frac{\partial V}{\partial s} + \frac{1}{\rho}\frac{\partial p}{\partial s} = 0 \tag{15}$$

If ρ is a function of p, integrating of equation (15) along the streamline AB yields

$$\frac{1}{2}q^2 + V + \int \frac{dp}{\rho} = C \tag{16}$$

$\square$

All the best...
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