Simplex Method
(M.Sc. Sem-III)

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WOLFE'S MODIFIED SIMPLEX METHOD

The iterative procedure for the solution of a quadratic programming problem by Wolfe's modified simplex method can be summarized as follows :

Step-1 : Convert the inequality constraints into equations by introducing the slack varibales S_t^2 in the ith constraint, i = 1, 2, ..., m, and the slack variables S_{m+j}^2 in the jth non-negativity constraint j = 1, 2, ..., n.

Step-2 : Construct the Langrangian function

$$L(x, S, \lambda) = f(x) - \sum_{i=1}^{m} \lambda_i \left[\sum_{j=1}^{n} a_{ij} x_j - b_i + S_i^2 \right] - \sum_{j=1}^{n} \lambda_{m+j} \left(-x_j + S_{m+j}^2 \right)$$

where $x = (x_1, x_2, ..., x_n)$, $S = (S_1, ..., S_{m+n})$, $\lambda = (\lambda_1, ..., \lambda_{m+n})$.

Differentiate $L(x, S, \lambda)$ partially with respect to the components of x, S and λ and equate the first order partial derivatives equal to zero. Derive the Kuhn-Tucker conditions from the resulting equations.

Step-3 : Introduce the non-negative artificial variables A_j, $j = 1, 2, ..., n$ in the Kuhn-Tucker condition.

$$c_j + \sum_{k=1}^{n} d_{jk} x_k - \sum_{i=1}^{m} \lambda_i a_{ij} + \lambda_{m+j} = 0$$

for j = 1, 2, ..., n and construct an objective function

$z = A_1 + A_2 + ... + A_n$

Step-4 : Obtain an initial basic feasible solution to the LPP :

Minimize $z = A_1 + A_2 + ... + A_n$ subject to the constraints :

$$\sum_{k=1}^{n} d_{jk} x_k - \sum_{i=1}^{m} \lambda_i a_{ij} + \lambda_{m+j} + A_j = -c_j \qquad (j = 1, 2, ..., n)$$

$$\sum_{j=1}^{n} a_{ij} x_j + x_{n+i} = b_i \qquad (i = 1, 2, ..., m)$$

$$A_j, \lambda_i, \lambda_{m+j}, x_j \geq 0 \qquad (i = 1, 2, ..., m;\ j = 1, 2, ..., n)$$

where $x_{n+1} = S_i^2$, $i = 1, 2, ..., m$, and satisfying the complementary slackness conditions.

$$\sum_{j=1}^{n} \lambda_{m+j} x_j + \sum_{i=1}^{m} x_{n+i} \lambda_i = 0$$

Step-5 : Use two phase simplex method to obtain an optimum solution to the LPP of step-4, the solution satisfying the complementary slackness condition.

Step-6 : The optimum solution obtained in step-5 is an optimum solution to the given QPP also.

Note : If the given QPP is in the minimization form, convert it into that of maximization by appropriate adjustment in $f(x_1, x_2, ..., x_n)$.

SAMPLE PROBLEM

1. Use Wolfe's method to solve the QPP :

$$\textit{Maximize } z = 2x_1 + 3x_2 - 2x_1^2 \text{ subject to the constraints :}$$

$$x_1 + 4x_2 \le 4,\ x_1 + x_2 \le 2$$

$$x_1,\ x_2 \ge 0\,.$$

Sol. We convert inequality constraints into equations by introducing slack variables S_1^2 and S_2^2 respectively. Considering $x_1 \ge 0$ and $x_2 \ge 0$ also as the inequality constraints, we convert them also into equations by introducing slack variables S_3^2 and S_4^2 in them. The problem thus becomes :

$$\textit{Maximize } z = 2x_1 + 3x_2 - 2x_1^2 \text{ subject to the constraints :}$$

$$x_1 + 4x_2 + S_1^2 = 4,\ x_1 + x_2 + S_2^2 = 2,\ -x_1 + S_3^2 = 0,\ -x_2 + S_4^2 = 0.$$

Construct the Lagrangian function

$$L = L(x_1,\ x_2,\ S_1,\ S_2,\ S_3,\ S_4,\ \lambda_1,\ \lambda_2,\ \lambda_3,\ \lambda_4)$$
$$= (2x_1 + 3x_2 - 2x_1^2) - \lambda_1(x_1 + 4x_2 + S_1^2 - 4) - \lambda_2(x_1 + x_2 + S_2^2 - 2) - \lambda_3(-x_1 + S_3^2) - \lambda_4(-x_2 + S_4^2).$$

As x_1^2 represents a negative semi-definite quadratic form $z = 2x_1 + 3x_2 - 2x_1^2$ is concave in $x_1,\ x_2$. Thus, maxima of L will be maxima of $z = 2x_1 + 3x_2 - 2x_1^2$ and *vice-versa*. To derive the necessary and sufficient conditions for maxima of L (and hence of z) we equate the first-order partial derivatives of L w.r.t. the variables $x_1,\ x_2,\ S_i^{'}{'s}$ and $\lambda_i{'s}$. Thus, we have

$$\frac{\partial L}{\partial x_1} = 2 - 4x_1 - \lambda_1 - \lambda_2 + \lambda_3 = 0 \qquad \frac{\partial L}{\partial x_2} = 3 - 4x_1 - \lambda_2 + \lambda_4 = 0$$

$$\frac{\partial L}{\partial S_1} = -2\lambda_1 S_1 = 0; \qquad \frac{\partial L}{\partial \lambda_1} = x_1 + 4x_2 + S_1^2 - 4 = 0$$

$$\frac{\partial L}{\partial S_2} = -2\lambda_2 S_2 = 0; \qquad \frac{\partial L}{\partial \lambda_2} = x_1 + x_2 + S_2^2 - 2 = 0$$

$$\frac{\partial L}{\partial S_3} = -2\lambda_3 S_3 = 0; \qquad \frac{\partial L}{\partial \lambda_3} = -x_1 + S_3^2 = 0$$

$$\frac{\partial L}{\partial S_4} = -2\lambda_4 S_4 = 0; \qquad \frac{\partial L}{\partial \lambda_4} = -x_2 + S_4^2 = 0$$

Upon simplification and necessary manipulations these yield :

$$(1) \quad \begin{cases} 4x_1 + \lambda_1 + \lambda_2 - \lambda_3 = 2,\ 4\lambda_1 + \lambda_2 - \lambda_4 = 3 \\ \quad x_1 + 4x_2 + S_1^2 = 4,\ x_1 + x_2 + S_2^2 = 2 \end{cases}$$

$$(2) \quad \lambda_1 S_1^2 + \lambda_2 S_2^2 + x_1\lambda_3 + x_2\lambda_4 = 0,\ x_1,\ x_2,\ S_1^2,\ S_2^2,\ \lambda_i \ge 0,\ i = 1,\ 2,\ 3,\ 4.$$

A solution $x_j,\ j = 1,\ 2$ to (1) above and satisfying (2) shall necessarily be an optimal one for maximizing L. To determine the solution to the above simultaneous equation (1), we introduce the artificial variables A_1 and A_2 (both non-negative) in the first two constraints of (1) and construct the dummy objective function $z = A_1 + A_2$.

Then the problem becomes

Maximize $z = -A_1 - A_2$ subject to the constraints:

$$4x_1 + \lambda_1 + \lambda_2 - \lambda_3 + A_1 = 2$$
$$4\lambda_1 + \lambda_2 - \lambda_4 + A_2 = 3$$
$$x_1 + 4x_2 + x_3 = 4$$
$$x_1 + x_2 + x_4 = 2$$
$$x_1,\ x_2,\ x_3,\ x_4 \geq 0.$$
$$A_1,\ A_2,\ \lambda_i \geq 0,\ i = 1, \ldots, 4$$

satisfying the complementary slackness condition $\Sigma\lambda_i x_i = 0,$ where we have replaced S_1^2 by x_3 and S_2^2 by x_4.

The optimum solution to the above LPP shall now be obtained by the two phase simplex method. An initial basic feasible solution to the LPP is clearly given by :

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} A_1 \\ A_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 2 \\ 3 \\ 4 \\ 2 \end{bmatrix}$$

The simplex iterations leading to an optimum solution are :

Initial Iteration. Enter x_1 and drop A_1.

c_B	y_B	x_B	x_1	x_2	x_3	x_4	λ_1	λ_2	λ_3	λ_4	A_1	A_2
1	A_1	2	**4**	0	0	0	1	1	−1	0	1	0
1	A_2	3	0	0	0	0	4	1	0	−1	0	1
0	x_3	4	1	4	1	0	0	0	0	0	0	0
0	x_4	2	1	1	0	1	0	0	0	0	0	0
	z	5	4	0	0	0	5	2	−1	−1	0	0

From the above table, we observe that x_1, λ_1 *or* λ_2 can enter the basis. But λ_1 and λ_2 will not enter the basis, because x_3 and x_4 are in the basis. This is in view of the complimentary slackness conditions $\lambda_1 x_3 = 0$ and $\lambda_2 x_4 = 0.$

First Iteration. Enter x_2 and drop x_3.

c_B	y_B	x_B	x_1	x_2	x_3	x_4	λ_1	λ_2	λ_3	λ_4	A_2
0	x_1	1/2	1	0	0	0	1/4	1/4	1/4	0	0
1	A_2	3	0	0	0	0	4	1	0	−1	1
0	x_3	7/2	0	**4**	1	0	−1/4	−1/4	1/4	0	0
0	x_4	3/2	0	1	0	1	−1/4	−1/4	1/4	0	0
	z	3	0	0	0	0	4	1	0	0	0

Here, we observe that either λ_1 or λ_2 can enter the basis. But x_3 and x_4 are still in the basis, therefore these cannot enter the basis because of the complementary slackness conditions. However, since λ_4 is not in the basis, x_2 can enter the basis (because of the condition $\lambda_4 x_2 = 0$).

Second Iteration. Enter λ_1 and drop A_2.

c_B	y_B	x_B	x_1	x_2	x_3	x_4	λ_1	λ_2	λ_3	λ_4	A_2
0	x_1	1/2	1	0	0	0	1/4	1/4	−1/4	0	0
1	A_2	3	0	0	0	0	**4**	1	0	−1	1
0	x_2	7/8	0	1	1/4	0	−1/16	−1/16	1/16	0	0
0	x_4	5/8	0	1	−1/4	1	−3/16	−3/16	3/16	0	0
	z	3	0	0	0	0	4	1	0	−1	0

From this table, we see that λ_1 or λ_2 can enter the basis. But since x_4 is in the basis, λ_2 can't enter the basis and hence λ_1 enters the basis.

Final Iteration. Optimum solution.

c_B	y_B	x_B	x_1	x_2	x_3	x_4	λ_1	λ_2	λ_3	λ_4
0	x_1	5/6	1	0	0	0	0	3/16	−1/4	1/16
0	λ_1	3/4	0	0	0	0	1	1/4	0	−1/4
0	x_2	59/64	0	1	1/4	0	0	−3/64	1/16	−1/64
0	x_4	49/64	0	0	−1/4	1	0	−9/64	3/16	−3/64
	z	0	0	0	0	0	0	0	0	0

The optimum solution is

$$x_1 = 5/16,\ x_2 = 59/64 \text{ and maximum } z = 3.19$$

PROBLEMS

Use :Wolfe's method in solving the following quadratic programming problems :

1. Maximize $z = 4x_1 + 6x_2 - 2x_1^2 - 2x_1x_2 - 2x_2^2$ subject to the constraints :

$$x_1 + 2x_2 \le 2,\ x_1,\ x_2 \ge 0.$$

2. Maximize $z = 8x_1 + 10x_2 - x_1^2 - x_2^2$ subject to the constraints :

$$3x_1 + 2x_2 \le 6,\ x_1,\ x_2 \ge 0.$$

3. Maximize $z = 6x_1 + 3x_2 - 4x_1x_2 - 2x_1^2 - 3x_2^2$ subject to the constraints :

$$x_1 + x_2 \le 1,\ 2x_1 + 3x_2 \le 4;\ x_1,\ x_2 \ge 0.$$