M.S c Mathematics –SEM 3 Functional Analysis- CC-11 Unit 3

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Dual Space or (conjugate space)

Let E be a normed linear space over a field K (which is R or C) .Then the set B(E,K) of all continuous linear functional on E is a Baanach space with respect to pointwise linear operations and the norm defined by

$$||T|| = \sup_{\substack{x \in E \\ x \neq 0}} \frac{|T(x)|}{||x||}, T \in B(E,F)$$

The set B(E,K) is denoted by E^* and is called the (topological) dual space or the conjugate space or the adjoint space of E (Every element of $E^*$is a continuous linear functional on E).Thus the dual space of every normed linear space is a Banach space.

Example

If T is a continuous linear transformation of a normed linear space E into a normed linear space f, an if M is its null space (i.e kernel) show that T

induces a natural linear transformation T' of $\frac{E}{M}$ into F and that

$$||T'|| = ||T||$$

Proof

We know that $M = \{x \in E: Tx = 0\} = T^{-1}\{0\}$ *is* **a closed linear subspace of E and hence $\frac{E}{M}$ is a normed linear space with the norm of a coset X+ M in $\frac{E}{M}$ is defined by**

$$||x + M|| = \inf_{v \in M} ||x + v||$$

We define $T': \frac{E}{M} \to F$ **by** $T'(x + M) = Tx$

Let x+M ,y+M, be any two elements of $\frac{E}{M}$ and α and β be any scalars .Then

$$T'\{\alpha(x + M) + \beta(y + M)$$

$$= T'((\alpha x + M) + (\beta y + M))$$

$$= T'((\alpha x + \beta y) + M$$

$$= T(\alpha x + \beta y)$$

$$= \alpha Tx + \beta Ty$$

$$= \alpha T'(x + M) + \beta T'(y + M)$$

Hence T' is a linear transformation

Here we have $x \in E, v \in M \Rightarrow x + v \in E$ **and also every** $x \in E$ **can be written as** $x + 0$ **with** $0 \in M$.

$$||T'|| = sup\{||T'(x + M)||: x \in E, ||x + M|| \leq 1\}$$

$$= sup\{||Tx||: x \in E, \inf_{v \in M} ||x + v|| \leq 1$$

$$= sup\{||Tx||: x \in E, v \in M, ||x + v|| \leq 1$$

$$= \boldsymbol{sup}\{||\boldsymbol{Tx} + \boldsymbol{Tv}||: \boldsymbol{x} \in \boldsymbol{E} \text{ }, \boldsymbol{v} \in \boldsymbol{M}, ||\boldsymbol{x} + \boldsymbol{v}|| \leq \mathbf{1}$$

$$[\boldsymbol{T}(\boldsymbol{v}) = \mathbf{0} \text{ } \boldsymbol{for} \text{ } \boldsymbol{v} \in \boldsymbol{M}]$$

$$= \boldsymbol{sup} \text{ } \{||\boldsymbol{T}(\boldsymbol{x} + \boldsymbol{v})||: \boldsymbol{x} \in \boldsymbol{E}, \boldsymbol{v} \in \boldsymbol{M}, ||\boldsymbol{x} + \boldsymbol{v}|| \leq \mathbf{1}\}$$

$$= \boldsymbol{sup}\{||\boldsymbol{Tx}||: \boldsymbol{x} \in \boldsymbol{E}, ||\boldsymbol{x}|| \leq \mathbf{1}$$

$= ||\boldsymbol{T}||$ **Proved**