

e-content (lecture-14)

by

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MATH SEM-3 CC-11 UNIT-4 (Functional Analysis)

Topic: The Adjoint Operator.

Theorem: The adjoint operation $T \rightarrow T^*$ on $B(H)$ has the following properties:

$$(1) (T_1 + T_2)^* = T_1^* + T_2^* ,$$

$$(2) (\alpha T)^* = \bar{\alpha} T^* ,$$

$$(3) (T_1 T_2)^* = T_2^* T_1^* ,$$

$$(4) T^{**} = T ,$$

$$(5) \|T^*\| = \|T\| ,$$

$$(6) \|T^* T\| = \|T\|^2 .$$

Proof: we have for every $x, y \in H$

$$(x, (T_1 + T_2)^* y) = ((T_1 + T_2)x, y)$$

$$\begin{aligned}
&= (T_1x + T_2x, y) \\
&= (T_1x, y) + (T_2x, y) \\
&= (x, T_1^*y) + (x, T_2^*y) \\
&= (x, T_1^*y + T_2^*y) \\
&= (x, (T_1^* + T_2^*)y)
\end{aligned}$$

By uniqueness of adjoint operator we get

$$(T_1 + T_2)^* = T_1^* + T_2^* .$$

(2) we have for every $x, y \in H$

$$\begin{aligned}
(x, (\alpha T)^*y) &= ((\alpha T)x, y) = (\alpha(Tx), y) \\
&= \alpha(Tx, y) = \alpha(x, T^*y) = (x, (\overline{\alpha}T^*)y)
\end{aligned}$$

By uniqueness of adjoint operator we get

$$(\alpha T)^* = \overline{\alpha}T^*.$$

(3) we have for every $x, y \in H$

$$\begin{aligned}
(x, (T_1T_2)^*y) &= ((T_1T_2)x, y) \\
&= (T_1(T_2x), y) \\
&= (T_2x, T_1^*y) \\
&= (x, T_2^*(T_1^*y))
\end{aligned}$$

$$= (x, (T_2^* T_1^*)y)$$

By uniqueness of adjoint operator we get

$$(T_1 T_2)^* = T_2^* T_1^*.$$

(4) we have for every $x, y \in H$

$$\begin{aligned} (x, T^{**}y) &= (x, (T^*)^*y) = (T^*x, y) \\ &= \overline{(y, T^*x)} \\ &= \overline{(Ty, x)} = (x, Ty) \end{aligned}$$

By uniqueness of adjoint operator we get

$$T^{**} = T.$$

(5) we have for every $y \in H$

$$\begin{aligned} \|T^*y\|^2 &= (T^*y, T^*y) \\ &= (TT^*y, y) \\ &\leq \|TT^*y\| \|y\| \\ &\leq \|T\| \|T^*y\| \|y\| \\ \|T^*y\| &\leq \|T\| \|y\| \quad \text{for all } y \in H. \end{aligned}$$

$$\frac{\|T^*y\|}{\|y\|} \leq \|T\| \quad \text{for all } y \in H \text{ with } y \neq 0$$

$$\text{Sup}\left\{\frac{\|T^*y\|}{\|y\|} : y \in H, y \neq 0\right\} \leq \|T\|$$

Applying the result (i) to T^* in place of T we have

$$\begin{aligned} & \|(T^*)^*\| \leq \|T^*\| \\ \Rightarrow & \|T^{**}\| \leq \|T^*\| \\ \Rightarrow & \|T\| \leq \|T^*\| \dots\dots (\text{ii}) \end{aligned}$$

From (i) and (ii) we get

$$\|T^*\| = \|T\|.$$

(6) we have

$$\begin{aligned}
 \|T^*T\| &\leq \|T^*\| \|T\| \\
 &\leq \|T\| \|T\| \quad [\text{Since } \|T^*\| = \|T\|] \\
 &\leq \|T\|^2 \quad \dots\dots(\text{iii})
 \end{aligned}$$

we have for every $x \in H$

$$\begin{aligned} \|Tx\|^2 &= (Tx, Tx) \\ &= (T^*Tx, x) \\ &\leq \|T^*Tx\| \|x\| \\ &\leq \|T^*T\| \|x\| \|x\| \end{aligned}$$

$$\|Tx\|^2 \leq \|T^*T\| \|x\|^2 \text{ for every } x \in H \dots \text{(iv)}$$

$$\text{Now } \|T\| = \text{Sup}\{\|Tx\| : \|x\| \leq 1\}$$

$$\begin{aligned} \text{So } \|T\|^2 &= [\text{Sup}\{\|Tx\| : \|x\| \leq 1\}]^2 \\ &= \text{Sup}\{\|Tx\|^2 : \|x\| \leq 1\} \end{aligned}$$

From (iv) we see that if $\|x\| \leq 1$

$$\text{Then } \|Tx\|^2 \leq \|T^*T\|$$

$$\text{So } \text{Sup}\{\|Tx\|^2 : \|x\| \leq 1\} \leq \|T^*T\|$$

$$\|T\|^2 \leq \|T^*T\| \dots \text{(v)}$$

From (iii) and (v) we get

$$\|T^*T\| = \|T\|^2 .$$

END