KARUSH-KUHN-TUCKER (KKT) CONDITIONS[2]
(M.Sc. Sem-III)

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1. INEQUALITY CONSTRAINTS - KARUSH-KUHN-TUCKER (KKT) CONDITIONS[2]

This section extends the Lagrangean method to problems with inequality constraints. The main contrivbution of the section is the development of the general Karush-Kuhn Tucker (KKT) necessary conditions for determining the stationary points. These conditions are also sufficient under certain rules that will be stated later.

Consider the problem

$$\text{Maximize } z = f(X)$$

subject to

$$g(X) \leq 0$$

The inequality constraints may be converted into equations by using non-negative slack variables. Let $S_1^2 (\geq 0)$ be the slack quantity added to the ith constraint $g_i(X) \leq 0$ and define

$$S = (S_1, S_2, ..., S_m)^T, \; S^2 = (S_1^2, S_2^2, ..., S_m^2)^T$$

where m is the total number of inequality constraints. The Lagrangean function is thus given by :

$$L(X, S, \lambda) = f(x) - \lambda\left[g(X) + S^2\right]$$

Given the constraints

$$g(X) \leq 0$$

a necessary condition for optimality is that λ be non-negative (non-positive) for maximization (minimization) problems. This result is justified by noting that the vector λ measures the rate of variation of f with respect to g – that is,

$$\lambda = \frac{\partial f}{\partial g}$$

In the maximization case, as the right-hand side of the constraint $g(X) \leq 0$ increases from 0 to the vector ∂g, the solution space becomes less constrained and hence f cannot decrease, meaning that $\lambda \geq 0$. Similarly for minimization, as the right-hand side of the constraints increases, f cannot increase, which implies that $\lambda \leq 0$. If the constraints are equalities, that is, $g(X) = 0$, then λ becomes unrestricted in sign.

The restrictions on λ holds as part of the KKT necessary conditions. The remaining conditions will now be developed.

Taking the partial derivatives of L with respect to X, S and λ, we obtain

$$\frac{\partial L}{\partial X} = \nabla f(X) - \lambda \nabla g(X) = 0$$

$$\frac{\partial L}{\partial S_i} = -2\lambda_i S_i = 0, \; i = 1, 2, ..., m$$

$$\frac{\partial L}{\partial \lambda} = -\left(g(X) + S^2\right) = 0$$

The second set of equations reveals the following results :

1. If $\lambda_i \neq 0$, then $S_i^2 = 0$, which means that the corresponding resource is scarce, and, hence, it is consumed completely (equality constraint).

2. If $S_i^2 > 0$, then $\lambda_i = 0$. This means resource *i* is not scarce and, consequently, it has no affect on the value of *f* (i.e., $\lambda_i = \frac{\partial f}{\partial g_i} = 0$)

From the second and third sets of equations, we obtain

$\lambda_i g_i(X) = 0,\ i = 1,\ 2,\ \ldots..,\ m$

This new condition essentially repeats the foregoing argument, because if $\lambda_i > 0$,

$g_i(X) = 0$ or $S_i^2 = 0$; and if $g_i(X) < 0,\ S_i^2 > 0$, and $\lambda_i = 0$.

The KKT necessary conditions for maximization problem are summarized as :

$$\lambda \geq 0$$
$$\nabla f(X) - \lambda \nabla g(X) = 0$$
$$\lambda_i g_i(X) = 0, \quad i = 1,\ 2,\ \ldots.,\ m$$
$$g(X) \leq 0$$

These conditions apply to the minimization case as well, except that λ must be non-positive (verify!). In both maximization and minimization, the Lagrange multipliers corresponding to equality constraints are unrestricted in sign.

Sufficiency of the KKT Conditions. The Kuhn-Tucker necessary conditions are also sufficient if the objective function and the solution space satisfy specific conditions.

These conditions are summarized in Table-1.

It is simpler to verify that a function is convex or concave than to prove that a solution space is a convex set. For this reason, we provide a list of conditions that are easier to apply in practice in the sense that the convexity of the solution can be established by checking the convexity or concavity of the constraint functions. To provide these conditions, we define the generalized non-linear problems as

Maximize or minimize $z = f$ (X)

subject to

$$g_i(X) \leq 0,\ i = 1,\ 2,\ \ldots.,\ r$$
$$g_i(X) \geq 0,\ i = r+1,\ \ldots.,\ p$$
$$g_i(X) = 0,\ i = p+1,\ \ldots..,\ m$$

$$L(X,\ S,\ \lambda) = f(X) - \sum_{i=1}^{r} \lambda_i \left[g_i(X) + S_1^2 \right] - \sum_{i=r+1}^{p} \lambda_i \left[g_i(X) - S_1^2 \right] - \sum_{i=n+1}^{m} \lambda_i g_i(X)$$

where λ_i is the Lagrangean multiplier associated with constraint *i*. The conditions for establishing the sufficiency of the KKT conditions are summarized in Table-2.

The conditions in Table-2 represent only a subset of the conditions in Table-1 because a solution space may be convex without satisfying the conditions in Table-2.

TABLE-1

Sense of optimization	Required conditions	
	Objective function	Solution space
Maximization	Concave	Convex set
Minimization	Convex	Convex set

TABLE-2

Sense of optimization	Required conditions			
	$f(X)$	$g_i(X)$	λ_i	
Maximization	Concave	Convex	≥ 0	$(1 \leq i \leq r)$
		Concave	≤ 0	$(r+1 \leq i \leq p)$
		Linear	Unrestricted	$(p+1 \leq i \leq m)$
Minimization	Convex	Convex	≤ 0	$(1 \leq i \leq r)$
		Concave	≥ 0	$(r+1 \leq i \leq p)$
		Linear	Unrestricted	$(p+1 \leq i \leq m)$

Table-2 is valid because the given conditions yield a concave Lagrangean function $L(X, S, \lambda)$ in case of maximization and a convex $L(X, S, \lambda)$ in case of minimization. This result is verified by noticing that if $g_i(x)$ is convex, then $\lambda_i g_i(x)$ is convex if $\lambda_i \geq 0$ and concave if $\lambda_i \leq 0$. Similar interepretations can be established for all the remaining conditions. Observe that a linear function is both convex and concave. Also, if a function f is concave, then $(-f)$ is convex, and vice-versa.

Example-1

Consider the following minimization problem :

$$\text{Minimize } f(X) = x_1^2 + x_2^2 + x_3^2$$

subject to

$$g_1(X) = 2x_1 + x_2 - 5 \leq 0$$

$$g_2(X) = x_1 + x_3 - 2 \leq 0$$

$$g_3(X) = 1 - x_1 \leq 0$$

$$g_4(X) = 2 - x_2 \leq 0$$

$$g_5(X) = -x_2 \leq 0$$

This is a minimization problem, hence $\lambda \leq 0$. The KKT conditions are thus given as

$$(\lambda_1, \lambda_2, \lambda_3, \lambda_4, \lambda_5) \leq 0$$

$$(2x_1, 2x_2, 2x_3) - (\lambda_1, \lambda_2, \lambda_3, \lambda_4, \lambda_5)\begin{pmatrix} 2 & 1 & 0 \\ 1 & 0 & 1 \\ -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{pmatrix} = 0$$

$$\lambda_1 g_1 = \lambda_2 g_2 = ... = \lambda_5 g_5 = 0$$

$$g(X) \leq 0$$

These conditions reduce to

$$\lambda_1, \lambda_2, \lambda_3, \lambda_4, \lambda_5 \le 0$$
$$2x_1 - 2\lambda_1 - \lambda_2 + \lambda_3 = 0$$
$$2x_1 - \lambda_1 + \lambda_4 = 0$$
$$2x_3 - \lambda_2 + \lambda_5 = 0$$
$$\lambda_1(2x_1 + x_2 - 5) = 0$$
$$\lambda_2(x_1 + x_3 - 2) = 0$$
$$\lambda_3(1 - x_1) = 0$$
$$\lambda_4(2 - x_2) = 0$$
$$\lambda_5 x_3 = 0$$
$$2x_1 + x_2 \le 5$$
$$x_1 + x_3 \le 2$$
$$x_1 \ge 1,\ x_2 \ge 2,\ x_3 \ge 0$$

The solution is $x_1 = 1,\ x_2 = 2,\ x_3 = 0,\ \lambda_1 = \lambda_2 = \lambda_5 = 0,\ \ \lambda_3 = -2,\ \lambda_4 = -4.$ Because both $f(X)$ and the solution space $g(X) \le 0$ are convex, $L(X, S, \lambda)$ must be convex and the resulting stationary point yields a global constrained minimum.