e-content (lecture-12)

by

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MATH SEM-3 CC-11 UNIT-4 (Functional Analysis)

Topic:Frechet-Riesz representation theorem.

Theorem(Frechet-Riesz representation theorem for bounded linear functional in H)

Let f be a bounded (i.e continuous)linear functional on a Hilbert space H.Then there exists a unique vector y ∈H

such that $f(x) = (x, y) \quad \forall\, x \in$ H.

Proof: If f is zero functional then $f(x) = 0 \quad \forall\, x \in$ H

So we choose $y = 0$ then $(x, y) = 0$ and

hence $f(x) = (x, y) \quad \forall\, x \in$ H.

If f is not zero functional then its Kernel

M={ $x \in$ H: $f(x) = 0\}$ is a proper closed linear subspace of the Hilbert space H.

So there exists a non-zero vector $y_0 \in$ H such that y_0 is orthogonal to M .

Putting $= \frac{y_0}{\|y_0\|}$. Then z is orthogonal to M and $\|z\| = 1$. Now for all $x \in$ H, $f(x)z - f(z)x$ belongs M to M

Since $f(f(x)z - f(z)x) = f(x)f(z) - f(z)f(x) = 0$ and z is orthogonal to M hence

$$(f(x)z - f(z)x, z) = 0$$

$$\Rightarrow f(x).(z,z) - f(z)(x,z) = 0$$

$$\Rightarrow f(x).(z,z) = f(z)(x,z)$$

$$\Rightarrow \; f(x).\|x\|^2 = \left(x, \overline{f(z)}z\right)$$

$$\Rightarrow \; f(x) = \left(x, \overline{f(z)}z\right)$$

Taking $y = \overline{f(z)}z$ we get

$$f(x) = (x, y) \;\; \forall\, x \in \text{H}.$$

Uniqueness: Let y' be another vector in H such that

$$f(x) = (x, y') \;\; \forall\, x \in \text{H}.$$

Then $(x, y) = (x, y') \;\; \forall\, x \in \text{H}.$

Hence $(x, y) - (x, y') = 0 \ \forall \, x \in \mathrm{H}$.

$\Rightarrow \ (x, y - y' \,) = 0 \ \forall \, x \in \mathrm{H}$.

$\Rightarrow \ (y - y', y - y' \,) = 0$ Taking $x = y - y'$

$\Rightarrow \ y - y' = 0$

$\Rightarrow \ y = y'$ Hence y is a unique vector H .

END.