

e-content (lecture-10)

by

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MATH SEM-3 CC-11 UNIT-4 (Functional Analysis)

Topic: *Bessel's Inequality* .

Theorem : If $\{e_1, e_2, \dots, e_n\}$ be a finite orthonormal set in a Hilbert space H and x be any vector in H , then

$$(a) \sum_{i=1}^n |(x, e_i)|^2 \leq \|x\|^2$$

[called *Bessel's Inequality* for finite orthonormal set]

$$(b) x - \sum_{i=1}^n (x, e_i) e_i \perp e_j \text{ for each } j.$$

Proof of (a): Let $\alpha_i = (x, e_i)$ for $i = 1, 2, \dots, n$.

$$\text{Then } 0 \leq \|x - \sum_{i=1}^n \alpha_i e_i\|^2$$

$$= (x - \sum_{i=1}^n \alpha_i e_i, x - \sum_{i=1}^n \alpha_i e_i),$$

$$= (x, x) - \sum_{j=1}^n \bar{\alpha}_j (x, e_j) - \sum_{i=1}^n \alpha_i (e_i, x) +$$

$$\sum_{i=1}^n \sum_{j=1}^n \alpha_i \bar{\alpha}_j (e_i, e_j)$$

$$= (x, x) - \sum_{j=1}^n \bar{\alpha}_j \alpha_j - \sum_{i=1}^n \alpha_i \bar{\alpha}_i + \sum_{i=1}^n \alpha_i \left(\sum_{j=1}^n \bar{\alpha}_j (e_i, e_j) \right)$$

$$= \|x\|^2 - \sum_{j=1}^n |\alpha_j|^2 - \sum_{i=1}^n |\alpha_i|^2 + \sum_{i=1}^n \alpha_i \bar{\alpha}_i$$

[since $(e_i, e_j) = 0$ for $i \neq j$]

$$= \|x\|^2 - \sum_{j=1}^n |\alpha_j|^2$$

$$\Rightarrow \|x\|^2 \geq \sum_{j=1}^n |\alpha_j|^2$$

$$\Rightarrow \sum_{i=1}^n |(x, e_i)|^2 \leq \|x\|^2$$

(b) now we have

$$(x - \sum_{i=1}^n (x, e_i) e_i, e_j) = (x, e_j) - \sum_{i=1}^n (x, e_i) (e_i, e_j)$$

$$= (x, e_j) - (x, e_j)$$

[since $(e_i, e_j) = 0$ for $i \neq j$]

$$= 0$$

Hence $x - \sum_{i=1}^n (x, e_i) e_i \perp e_j$ for each j .

END.