# Rotational and irrotational fluid flow (6)

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## 1 The velocity potential or velocity function

Suppose that the fluid velocity at time $t$ is $\boldsymbol{q} = (u, v, w)$. Further consider that at the instant $t$, there exist a scalar function $\phi(x, y, z, t)$ uniform throughout the entire field of flow and such that

$$-d\phi = q \cdot dr = udx + vdy + wdz$$

i.e.,

$$-\Big(\frac{\partial \phi}{\partial x}dx + \frac{\partial \phi}{\partial y}dy + \frac{\partial \phi}{\partial z}dz\Big) = udx + vdy + wdz$$

$\implies$

$$u = -\frac{\partial \phi}{\partial x}, \quad v - \frac{\partial \phi}{\partial y}, \quad w = -\frac{\partial \phi}{\partial z}$$

$\therefore$

$$\boldsymbol{q} = -\nabla\phi = \text{grad}\phi \tag{1}$$

$\phi$ is called the velocity potential. The negative sign in (1) is a convention. It ensure that the flow take place from the higher to lower potential.

The necessary and sufficient condition for (1) to hold is

$$\boldsymbol{q} \times dr = 0 \qquad \text{i.e.,} \quad \text{curl } \boldsymbol{q} = 0$$

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or,

$$i\Big(\frac{\partial w}{\partial y}-\frac{\partial v}{\partial z}\Big)+j\Big(\frac{\partial u}{\partial z}-\frac{\partial w}{\partial x}\Big)+k\Big(\frac{\partial v}{\partial x}-\frac{\partial u}{\partial y}\Big)=0 \tag{2}$$

**Remark 1.** *The surface*

$$\phi(x,y,z,t)=constant \tag{3}$$

*are called equipotentials. The streamlines motion*

$$\frac{dx}{u}=\frac{dy}{v}=\frac{dz}{w} \tag{4}$$

*are cut at right angles by the surface given by the differential equation*

$$udx+vdy+wdz \tag{5}$$

*and the condition for the existence of such orthogonal surface is the condition that (5) may posses a solution of the form (4) at the considered instant t, the analytical condition being*

$$u\Big(\frac{\partial w}{\partial y}-\frac{\partial v}{\partial z}\Big)+v\Big(\frac{\partial u}{\partial z}-\frac{\partial w}{\partial x}\Big)+w\Big(\frac{\partial v}{\partial x}-\frac{\partial u}{\partial y}\Big)=0 \tag{6}$$

*when the velocity potential exists,(1) holds. Then*

$$\frac{\partial w}{\partial y}-\frac{\partial v}{\partial z}=\frac{\partial^2\phi}{\partial y\partial z}-\frac{\partial^2\phi}{\partial z\partial y}=0 \qquad \Longrightarrow \qquad \frac{\partial w}{\partial y}=\frac{\partial v}{\partial z}$$

*Similarly*

$$\frac{\partial u}{\partial z}=\frac{\partial w}{\partial x} \quad and \quad \frac{\partial v}{\partial x}=\frac{\partial u}{\partial y}$$

**Remark 2.** *When equation (1) holds, then flow is known as potential kind. It also known as motion is irrational. For such flow the field of* $\boldsymbol{q}$ *is conservative.*

**Remark 3.** *The equation of continuity of an incompressible fluid*

$$\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}+\frac{\partial w}{\partial z}=0 \tag{7}$$

*Suppose that the fluid move irrationally. Then the velocity potential* $\phi$ *exists such that*

$$u=-\frac{\partial\phi}{\partial x}, \quad v-\frac{\partial\phi}{\partial y}, \quad w=-\frac{\partial\phi}{\partial z} \tag{8}$$

*Using equations (7) and (8) reduces to*

$$\frac{\partial^2\phi}{\partial^2 x}+\frac{\partial^2\phi}{\partial^2 y}+\frac{\partial^2\phi}{\partial^2 z}=\nabla^2\phi=0 \tag{9}$$

*showing that* $\phi$ *is a harmonic function satisfying the Laplace equation* $\nabla^2\phi=0$*, where*

$$\nabla^2=\frac{\partial^2}{\partial^2 x}+\frac{\partial^2}{\partial^2 y}+\frac{\partial^2}{\partial^2 z} \tag{10}$$

## 1.1 The vorticity vector.

Let $\boldsymbol{q} = ui + vj + wk$ be the fluid velocity such that

$$\boldsymbol{q} \times dr \neq 0 \qquad \text{i.e.,} \quad \text{curl } \boldsymbol{q} \neq 0$$

Then the vector

$$\Lambda = \boldsymbol{q} \times dr = \text{curl } \boldsymbol{q} \tag{11}$$

is called vorticity vector.

Let $\Lambda_x$, $\Lambda_y$ and $\Lambda_z$ be the components of $\Lambda$ in Cartesian coordinate. Then (11) reduces to

$$\Lambda_x i + \Lambda_y j = \Lambda_z k = i\Big(\frac{\partial w}{\partial y} - \frac{\partial v}{\partial z}\Big) + j\Big(\frac{\partial u}{\partial z} - \frac{\partial w}{\partial x}\Big) + k\Big(\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}\Big) \tag{12}$$

so that

$$\Lambda_x = \frac{\partial w}{\partial y} - \frac{\partial v}{\partial z}, \quad \Lambda_y = \frac{\partial u}{\partial z} - \frac{\partial w}{\partial x}, \quad \Lambda_z = \frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}$$

### 1.1.1 Vortex line.

A vortex line is a curve drawn in the fluid such that the tangent to it every point is in the direction of the vorticity vector $\Lambda$.

Let $\Lambda = \Lambda_x i + \Lambda_y j + \Lambda_z k$ and let $r = xi + yj + zk$ be the position vector of a point $P$ on the vortex line. The $\Lambda$ is parallel to $dr$ at point $P$ on the vortex line. Hence the equation of vortex lines is given by

$$\Lambda \times dr = 0 \tag{13}$$

i.e.,

$$\Big(\Lambda_x i + \Lambda_y j + \Lambda_z k\Big) \times \Big(dxi + dyj + dzk\Big) = 0$$

or,

$$(\Lambda_y dz - \Lambda_z dy)i + (\Lambda_z dx - \Lambda_x dz)j + (\Lambda_x dy - \Lambda_y dx)k$$

$\Longrightarrow$

$$\Lambda_y dz - \Lambda_z dy = 0 \quad \Lambda_z dx - \Lambda_x dz = 0 \;\; \text{text} \;\; \Lambda_x dy - \Lambda_y dx$$

So that

$$\frac{dx}{\Lambda_x} = \frac{dy}{\Lambda_y} = \frac{dz}{\Lambda_z} \tag{14}$$

All the best...

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