Lamb's hydrodynamical equation (8)

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1 Deduction of Lamb's hydrodynamical equations

Equation of continuity for fluid flow

$$\frac{D}{Dt} = \frac{\partial}{\partial t} + \boldsymbol{q}\cdot\nabla \tag{1}$$

Euler's equation of continuity be

$$\frac{D\boldsymbol{q}}{Dt} = \boldsymbol{F} - \frac{1}{\rho}\nabla p \tag{2}$$

where $\boldsymbol{F}$ be external force. Let ρ and p be density of fluid and pressure respectively.

Then equation (2) can be written as

$$\frac{D\boldsymbol{q}}{Dt} = \frac{\partial \boldsymbol{q}}{\partial t} + \Big(\boldsymbol{q}\cdot\nabla\Big)\boldsymbol{q} = \boldsymbol{F} - \frac{1}{\rho}\nabla p \tag{3}$$

But

$$\nabla(\boldsymbol{q}\cdot\boldsymbol{q}) = 2\Big[\boldsymbol{q}\times \mathrm{curl}\boldsymbol{q}\Big]$$

so that

$$(\boldsymbol{q}\cdot\nabla)\boldsymbol{q} = \frac{1}{2}\nabla(\boldsymbol{q}\cdot\boldsymbol{q}) - \boldsymbol{q}\times \mathrm{curl}\boldsymbol{q}$$

or,

$$(\boldsymbol{q}\cdot\nabla)\boldsymbol{q} = \nabla(\frac{\boldsymbol{q}^2}{2}) + \mathrm{curl}\boldsymbol{q}\times\boldsymbol{q} \tag{4}$$

Then equation (3) can be written with the help of equation (4)

$$\frac{\partial \boldsymbol{q}}{\partial t} + \nabla(\frac{\boldsymbol{q}^2}{2}) + \mathrm{curl}\boldsymbol{q}\times\boldsymbol{q} = \boldsymbol{F} - \frac{1}{\rho}\nabla p \tag{5}$$

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If the motion of fluid be rotational, then vorticity vector $\mathbf{\Lambda}$ is given by

$$\mathbf{\Lambda} = \text{curl}\boldsymbol{q} \tag{6}$$

Then equation (5) becomes

$$\frac{\partial \boldsymbol{q}}{\partial t} + \nabla(\frac{\boldsymbol{q}^2}{2}) + \mathbf{\Lambda} \times \boldsymbol{q} = -\nabla V - \frac{1}{\rho}\nabla p \tag{7}$$

If $\boldsymbol{F}$ be conservative force, then there exists force potential V such that $\boldsymbol{F} = -\nabla V$.

Equation (7) is known as Lamb's hydrodynamical equation.

Example 1. *A mass of gravitating fluid is at rest under its own attraction only, the free surface being a sphere of radius b and the inner surface a rigid concentric shell of radius a. Show that if the shell suddenly disappears, the initial pressure at any point of the fluid at distance r from the center is ,*

$$\frac{2}{3}\pi\gamma\rho^2(b-a)(r-a)\left(\frac{a+b}{r}+1\right)$$

Solution 1. *Let t, v' be the velocity at a distance r' from the center and let the radius of the inner spherical cavity be r. Let p be the pressure at a distance r'. Then the equation of continuity is*

$$r'^2 v' = F(t) \tag{8}$$

$$\textit{from (8)}, \qquad \frac{dv'}{dt} = \frac{F'(t)}{r'^2} \tag{9}$$

$$\textit{Attraction at distance} \quad r' = \frac{(4/3) \times \pi\gamma\rho(r'^3 - r^3)}{r'^2}$$

where γ is the usual constant of gravitation.

Hence the equation of motion is

$$\frac{\partial v'}{\partial t} + v'\frac{\partial v'}{\partial r'} = -\frac{4}{3}\pi\gamma\rho\left(r' - \frac{r^3}{r'^2}\right) - \frac{1}{\rho}\frac{\partial p}{\partial r'}$$

$$i.e, \frac{F'(t)}{r'^2} + \frac{\partial}{\partial r'}\left(\frac{1}{2}v'2\right) = -\frac{4}{3}\pi\gamma\rho\left(r' - \frac{r^3}{r'^2}\right) - \frac{1}{\rho}\frac{\partial p}{\partial r'}$$

integrating with respect to r',we obtain

$$-\frac{F'(t)}{r'} + \frac{1}{2}v^2 = -\frac{4}{3}\pi\gamma\rho\left(\frac{r'^2}{2} + \frac{r^3}{r'}\right) - \frac{p}{\rho} + C \quad \textit{C being and arbitrary constant} \tag{10}$$

Initially, when $t = 0$, then $v' = 0$, and $p = P$ (say). Hence (10) yields

$$-\frac{F'(0)}{a} = -\frac{4}{3}\pi\gamma\rho\left(\frac{r'^2}{2} + \frac{a^3}{r'}\right) - \frac{P}{\rho} + C \tag{11}$$

But, $P = 0$ when $r' = a$ and $r' = b$ so equation (11) gives

$$-\frac{F'(0)}{a} = -\frac{4}{3}\pi\gamma\rho\left(\frac{a^2}{2} + a^2\right) + C \tag{12}$$

$$\text{and} \qquad -\frac{F'(0)}{a} = -\frac{4}{3}\pi\gamma\rho\left(\frac{b^2}{2} + \frac{a^3}{b}\right) + C \tag{13}$$

subtracting (13) from (12), we have

$$F'(0)\left(\frac{1}{b} - \frac{1}{a}\right) = \frac{4}{3}\pi\gamma\rho\left\{\frac{(b^2 - a^2}{2} + a^2\left(\frac{a}{b} - 1\right)\right\}$$

$$F'(0)\frac{a-b}{ab} = \frac{4}{3}\pi\gamma\rho\left\{\frac{(b-a)(b+a)}{2} + \frac{a^2(a-b)}{b}\right\}$$

$$\text{or} \qquad F'(0) = -(2/3) \times \pi\gamma ab(a + b(4/3) \times \pi\gamma\rho a^3 \tag{14}$$

Multiplying (12) by a and (13) by b then subtracting, we get

$$0 = \frac{4}{3}\pi\gamma\rho\left(\frac{b^3}{2} - \frac{a^3}{2}\right) + C(a-b)$$

$$\text{or} \qquad C(b-a) = (2/3) \times \pi\gamma\rho(b-a)(b^2 + a^2 + ba)$$

$$C(a-b) = (2/3) \times \gamma\rho(b-a)(a^2 + b^2 + ab) \tag{15}$$

putting the values of $F'(0)$ and C in (11), we get

$$-\frac{1}{r'}\left\{-\frac{2}{3}\pi\gamma\rho(a+b) + \frac{4}{3}\pi\gamma\rho a^3\right\} = -\frac{4}{3}\pi\gamma\rho\left(\frac{r'^2}{2} + \frac{a^3}{r'}\right) - \frac{P}{\rho} + \frac{2}{3}\pi\gamma\rho\left(a^2 + b^2 + ab\right)$$

$$\therefore \qquad P = \frac{2}{3}\pi\gamma\rho^2\left\{a^2 + b^2 + ab - \frac{ab(a+b)}{r'} - r'^2\right\}$$

$$\text{or} \qquad P = \frac{2}{3}\pi\gamma\rho^2(r'-a)(b-r')\left(\frac{(a+b)}{r'} + 1\right) \tag{16}$$

For the required result, replace r' by r in (16)

All the best...
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