The energy equation and example (10)

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1 The energy equation

Statement 1. *The rate of change of total energy (kinetic, potential and intrinsic) of any portion of a compressible inviscid fluid as it moves about is equal to the rate at which work is being done by the pressure on the boundary. The potential due to the extraneous forces is supposed to be independent of time.*

Proof. Consider any arbitrary closed surface S drawn in the region occupied by the inviscid fluid and let V be the volume of the fluid within S. Let ρ be the density of the fluid particle P within S and dV be the volume element surrounding P. Let $q(r,t)$ be the velocity of P. Then, the Euler's equation of motion is

$$dq/dt = -(1/\rho)\nabla p + F \tag{1}$$

Let the external forces be conservative so that there exists a force potential Ω which is independent of time. Thus $\quad F = -\nabla\Omega \quad$ and $\quad \partial\Omega/\partial t = 0.$
Using the above results and then multiplying both side of (1) scalarly by q, we get

$$\rho\left(q.\frac{dq}{dt}\right) = -q.\nabla p - \rho\left[q.\nabla\Omega\right] \qquad or \qquad \rho\left[\frac{d}{dt}\left(\frac{1}{2}q^2\right) + (q.\nabla)\right] = -q.\nabla p \tag{2}$$

But $\quad \frac{d\Omega}{dt} = \frac{\partial\Omega}{\partial t} + (q.\nabla)\Omega, \quad$ since $\quad \frac{\partial\Omega}{\partial t} = 0$
Hence, equation (2) becomes

$$\rho\frac{d}{dt}\left(\frac{1}{2}q^2 + \Omega\right) = -q.\nabla p \tag{3}$$

since the elementary mass remains invariant thought the motion, so

$$d(\rho V)/dt = 0 \tag{4}$$

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Integrating both sides of (3) over V, we have

$$\int_V \frac{d}{dt}\left(\frac{1}{2}q^2\right)\rho dV + \int_v \frac{d}{dt}(\rho\Omega)dV = -\int_V (q.\nabla p)dV = -\int_V (q.\nabla p)dV$$

$$\text{or} \qquad \int_V \left\{\frac{d}{dt}\left(\frac{1}{2}q^2\right)\rho dV + \frac{1}{2}q^2\frac{d}{dt}(\rho dV)\right\} + \int_V \frac{d}{dt}(\rho\Omega dV) = -\int_V (q.dV)$$

$$\text{Thus,} \qquad \frac{d}{dt}\int_V \left(\frac{1}{2}\rho q^2\right)dV + \frac{d}{dt}\int_V (\rho\Omega)dv = -\int_V (q.\nabla p)dV \tag{5}$$

Let T, W and I denotes the kinetic, potential and intrinsic (internal) energies respectively. Then by definitions

$$T = \int_V \frac{1}{2}\rho q^2 dV, \qquad W = \int_V \rho\Omega dV, \qquad I = \int_V \rho E dV, \tag{6}$$

where E is the intrinsic energy per unit mass,

$$\text{since} \qquad \nabla.(pq) = p\nabla.q + q.\nabla p, \qquad \text{we have} \qquad q.\nabla p = \nabla.(pq) - p\nabla.q$$

$$\therefore \quad R.H.Sof(4) = -\int_V \nabla.(pq)dV + \int_V p\nabla.qdV = \int_s pq.nds + \int_s p\nabla.qdV, \tag{5}$$

when n is unit inward normal and ds is the element of the fluid surface S. We now prove that

$$\int_V p\nabla.qdV = -\frac{dI}{dt} \tag{5}$$

Now, E is defined as the work done by the unit mass of the fluid against external pressure P (assuming that there exists a relation between pressure and density) from its actual state to some standard state in which P_0 and ρ_0 are the values of pressure and density respectively.

$$\therefore \qquad E = \int_V^{V_0} PdV, \qquad where V\rho = 1 \qquad i.e., \qquad V = 1/\rho$$

$$\text{or} \qquad E = \int_\rho^{\rho_0} pd\left(\frac{1}{\rho}\right) = -\int_\rho^{\rho_0} \frac{p}{\rho^2}d\rho = \int_\rho^{\rho_0} \frac{p}{\rho^2}dp \tag{4}$$

$$\text{from (1),} \qquad \frac{dE}{d\rho} = \frac{p}{\rho^2} \qquad \text{and so} \qquad \frac{dE}{dt} = \frac{dE}{d\rho}\frac{d\rho}{dt} = \frac{p}{\rho^2}\frac{d\rho}{dt}$$

Multiplying both sides by ρdV and then integrating over a volume V, we have

$$\int_V \frac{dE}{dt}\rho dV = \int_V \frac{p}{\rho}\frac{d\rho}{dt}dV \tag{3}$$

$$\text{But} \qquad \frac{d}{dt}(E\rho dV) = \frac{dE}{dt}\rho dV + E\frac{d}{dt}(\rho dV)$$

$$\therefore \qquad \frac{d}{dt}(E\rho dV) = \frac{dE}{dt}\rho dV, \text{using } (4) \tag{2}$$

Also from the equation of continuity, $$\frac{d\rho}{dt} = -\rho\nabla.q \tag{2}$$

Using (1) and (1),(1) red uses to

$$\frac{d}{dt}\int_V E\rho dV = -\int_V p\nabla.qdV, \text{by}(6)$$

which proves (1).
Again the rate of work done by the fluid pressure on an element δS of S is $p\delta Sn.q$
Hence the net rate at which work is being done by the fluid pressure is

$$\int_S pq.nds = R, \qquad (say) \tag{2}$$

Using (1)and (1), (1) reduces to

$$\text{R.H.S of } (4) = R - dI/dt \tag{2}$$

Here using (6) and (1),(4) reduces to

$$\frac{d}{dt}(T + W + I) = R \tag{2}$$

which is the desired energy equation. It is also known as "the Volume integral from of Bernoulli's equation".
Re-writing the equation(1),

$$\frac{d}{dt}(T + W) = R - \frac{dI}{dt}\int_s pq.nds + \int_V p\nabla.qdV \tag{2}$$

corollary 1. ***Energy equation for incompressible fluids.***
since $I = 0$ for incompressible fluids, (1) reduce to

$$\frac{d}{dt}(T + W) = R \tag{2}$$

□

2 Example

Example 1. *An infinite mass of fluid is acted on by a force$\mu/r^{\frac{3}{2}}$ per unit mass directed to the origin. If initially the fluid is at rest and there is cavity in the form of the sphere $r = c$ in it, show that the cavity will be filled up after an interval of time $(2/5\mu)^{\frac{1}{2}}c^{\frac{5}{4}}$*

Solution 1. *At any time t, let v' be the velocity at distance r' from the center. Again, let r be the radius of the cavity and v its velocity.Then the equation of continuity yields*

$$r'^2 v' = r^2 v \tag{3}$$

when the radius of the cavity is r, then

$$\begin{aligned} \textit{Kinetic energy } &= \int_r^\infty \frac{1}{2}(4\pi r'^2 \rho dr').v^2 \\ &= 2\pi\rho r^4 v^2 \int_r^\infty \frac{dr'}{r'^2}, \textit{ using } 3) \\ &= 2\pi\rho r^3 v^2 \end{aligned}$$

The initial kinetic energy is zero.
Let V be the work function (or force potential) due to external forces. Then, we have

$$-\frac{\partial V}{\partial r'} = \frac{\mu}{r'^{3/2}} \qquad \textit{so that} \qquad V = \frac{2\mu}{r'^{1/2}}$$

$$\begin{aligned} \therefore \textit{ the work done} &= \int_r^c V dm, \qquad \textit{dm being the elementary mass} \\ &= \int_r^c \left(\frac{2\mu}{r'^{(1/2)}}\right) 4\pi r'^2 dr' \rho \\ &= 8\pi\mu\rho \int_r^c r'^{3/1} dr' \\ &= \frac{16}{5}\pi\rho\mu(c^{5/2} - r^{5/2}) \end{aligned}$$

We now use of energy equation, namely, *Increase in kinetic energy=work done*

$$\Rightarrow 2\pi\rho r^3 v^2 - 0 = (16/5) \times \pi\rho\mu(c^{5/2} - r^{5/2})$$

$$\therefore \qquad v = \frac{dr}{dt} = -\left(\frac{8\mu}{5}\right)^{1/2} \frac{(c^{5/2} - r^{5/2})^{1/2}}{r^{3/2}} \tag{4}$$

wherein negative sign is taken because r decreases as t increases
Let T be the time of filling up the cavity. Then (4) gives

$$\int_0^T dt = -\left(\frac{5}{8\mu}\right)^{1/2} \int_0^c \frac{r^{3/2} dr}{\sqrt{(c^{5/2} - r^{5/2})}}$$

$$T = -\left(\frac{5}{8\mu}\right)^{1/2} \int_0^c \frac{r^{3/2} dr}{\sqrt{(c^{5/2} - r^{5/2})}}$$

$$put r^{5/2} = c^{5/2} \sin^2\theta$$

$$\textit{so that} (5/2) \times r^{3/2} dr = 2c^{5/2} \sin\theta \cos\theta d\theta.$$

$$\therefore T = -\left(\frac{5}{8\mu}\right) \int_0^{\pi/2} \frac{4}{5} c^{5/4} \sin\theta d\theta = \left(\frac{2}{5\mu}\right)^{1/2} c^{5/4}.$$

Second Method. *Here the motion of the fluid will take place in such a manner so that each element of the fluid moves towards the center. Hence the free surface would be spherical. Thus the fluid velocity v' will be function of r' (the radial distance from the centre of the sphere which is taken as origin) and time t. Also, let v be the velocity at a distance r.*

Then the equation of continuity is

$$r'^2 v' = F(t) = r^2 v \tag{5}$$

from (5)

$$\frac{\partial v'}{\partial t} = \frac{F'(t)}{r'^2} \quad . \tag{6}$$

The equation of motion is

$$\frac{\partial v'}{\partial t} + v' \frac{\partial v'}{\partial r'} = -\frac{\mu}{r'^{3/2}} - \frac{1}{\rho}\frac{\partial p}{\partial r'};$$

or

$$\frac{F'(t)}{r'^2} + \frac{1}{2} v'^2 = -\frac{\mu}{r'^{3/2}} - \frac{1}{\rho}\frac{\partial p}{\partial r'}; \tag{7}$$

Integrating (7) with respect to r', we have

$$-\frac{F'(t)}{r'} + \frac{1}{2} v'^2 = \frac{2\mu}{r^{1/2}} - \frac{p}{\rho} + C \tag{8}$$

when $r' = \infty, v' = 0, p = 0$.

so from (8), $C = 0$. Then (8) becomes

$$-\frac{F'(t)}{r'} + \frac{1}{2} v'^2 = \frac{2\mu}{r'^{1/2}} - \frac{p}{\rho} \tag{9}$$

when $r' = r, v' = v, p = 0$.

so from (9), reduced to

$$-\frac{F'(t)}{r'} + \frac{1}{2} v'^2 = \frac{2\mu}{r^{1/2}} \tag{10}$$

$$\tag{11}$$

Now, $\quad (5) \Rightarrow F(t) r^2 v \quad \Rightarrow F'(t) = 2rv(dr/dt) + r^2(dv/dt)$

or $\quad F'(t) =$

All the best...
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