M.Sc Mathematics –SEM 2 Number Theory CC-10 Unit 4

E-content –By Dr Abhik Singh, Guest faculty, PG Department of Mathematics, Patna University, Patna

Content-Hurwitz Theorem

Theorem:

Let $\alpha=\frac{\sqrt{5}-1}{2}$ and $\beta>\sqrt{5}$ .Then the inequality

$|\alpha-\frac{x}{y}|<\frac{1}{\beta y^2}$

.... (1) has a finite number of solutions.

Proof : Let a fraction $\frac{a}{b}$ satisfy (1) so that we have

$$|\frac{\sqrt{5}-1}{2}-\frac{a}{b}|<\frac{1}{\beta b^2} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (2)$$

To prove the theorem we have to show that b can have only a finite number of different values. So the inequality (2) implies

$$\frac{1}{\beta b^2}\left(\frac{1}{\beta b^2}+\sqrt{5}\right)>\left|\frac{a}{b}-\frac{\sqrt{5}-1}{2}\right|\left(\left|\frac{a}{b}-\frac{\sqrt{5}-1}{2}\right|+|\sqrt{5}|\right)$$

$$\geq |\frac{a}{b} - \frac{\sqrt{5}-1}{2}||\frac{a}{b} - \frac{\sqrt{5}-1}{2} + \sqrt{5}|$$

$$= |\frac{a}{b} - \frac{\sqrt{5}-1}{2}||\frac{a}{b} + \frac{\sqrt{5}+1}{2}|$$

$$= |a^2+ab-b^2|/b^2$$

Here a^2+ab-b^2 is the product of irrational numbers. Hence it is not equal to zero .It follows that

$$\frac{1}{\beta b^2}\left(\frac{1}{\beta b^2} + \sqrt{5}\right) > \frac{1}{b^2} \text{ or}$$

$\frac{1}{\beta b^2} + \sqrt{5} > \beta$. .... (3)

Let $\beta = \sqrt{5} + \delta$ where δ is some positive real number. Then we have

$\frac{1}{(\sqrt{5}+\delta)b^2} + \sqrt{5} > \sqrt{5} + \delta$.

This implies $b^2 < \frac{1}{\delta(\sqrt{5}+\delta)}$= a finite number.

Thus we prove that b can have only a finite number of different values.

Second proof

Let the given inequality be satisfied by $\frac{a}{b}$ so that we have

$|\frac{\sqrt{5}-1}{2} - \frac{a}{b}| < \frac{1}{\beta b^2}$ ....(4)

To prove the theorem we have to show that b can have only a finite number of different values .Now , since $\beta>\sqrt{5}$ we can write (4) in the form $\frac{\sqrt{5}-1}{2}$ - $\frac{a}{b}$ = $\frac{\theta}{\sqrt{5}b}2$,when $|\theta|$ < 1

$\frac{\sqrt{5}b}{2}$ - $\frac{b}{2}$ - a=$\frac{\theta}{\sqrt{5}b}$

$\frac{\sqrt{5}b}{2}$ - $\frac{\theta}{\sqrt{5}b}$ = a +$\frac{b}{2}$

Squaring both sides we obtain

$5b^2/4+\theta^2/5b^2-\theta=a^2+ab+b^2/4$

Which reduces to

$a^2+ab-b^2=-\theta+\theta^2/5b^2$

Now $a^2+ab-b^2\leq|a^2+ab-b^2|=|-\theta+\theta^2/5b^2|$

Therefore $|a^2+ab-b^2|\leq|\theta|+\theta^2/5b^2|$ .... (5)

Let us assume that b has infinite number of different values which satisfy (4) .It follows that for sufficiently large value of b , $|\theta|+\theta^2/5b^2$ becomes a positive real number $\leq$1. Therefore for those values of b

$a^2+ab-b^2=0$ ....(6)

because a^2+ab-b^2 is necessarily an integer. Equation (6) implies

$4a^2+4ab+b^2 = 5b^2$

That is $2a+b=\sqrt{5}b$ which is impossible,$\sqrt{5}b$ is being irrational .Thus our assumption is proved. This complete the proof of the theorem.