Interior Point Algorithms
(M.Sc. Sem-III)

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1. Statement of the Problem

Karmakar's method requires the LP problem in the following form :

$$\text{Minimize } f = c^7 X$$

subject to

$$[a]X = 0$$

$$x_1 + x_2 + ... + x_n = 1 \qquad ....(i)$$

$$X \geq 0$$

where $X = \{x_1\ x_2....x_n\}^T$, $c = \{c_1\ c_2....c_n\}^T$, and $[a]$ is an $m \times n$ matrix. In addition, an interior feasible starting solution to Eq. (i) must be known. Usually, $X = \left\{\frac{1}{n}\ \frac{1}{n} .... \frac{1}{n}\right\}^T$ is chosen as the starting point. In addition, the optimum value of f must be zero for the problem. Thus

$$X^{(1)} = \left\{\frac{1}{n}\frac{1}{n}...\frac{1}{n}\right\}^T = \text{interior feasible.}$$

$$f_{\min} = 0 \qquad ....(ii)$$

Although most LP problems may not be available in the form of Eq. (i) while satisfying the conditions of Eq. (ii), it is possible to put any LP problem in a form that satisfies Eqs. (i) and (ii) as indicated below

2. Conversion of an LP Problem into the Required Form

Let the given LP problem be of the form :

$$\text{Minimize } d^T X$$

subject to

$$[\alpha]X = b \qquad ...(iii)$$

$$X \geq 0$$

To convert this problem into the form of Eq. (i), we use the procedure suggested in Ref. [] and define integers *m* and *n* such that X will be an (*n*–3) component vector and $[\alpha]$ will be a matrix of order $m-1 \times n-3$.

We now define the vector $\bar{z} = \{z_1\ z_2\ ....\ z_{n-3}\}^T$ as

$$\bar{z} = \frac{X}{\beta} \qquad ...(iv)$$

where β is a constant chosen to have a sufficiently large value such that

$$\beta > \sum_{i=1}^{n-3} x_i \qquad ...(v)$$

for any feasible solution X (assuming that the solution is bounded). By using Eq. (iv), the problem of Eq. (iii) can be stated as follows :

$$\text{Minimize } \beta d^T \bar{z}$$

subject to

$$[\alpha]\bar{z} = \frac{1}{\beta}b$$

$$\bar{z} \geq 0 \quad ...(vi)$$

We now define a new vector z as

$$z = \begin{Bmatrix} \bar{z} \\ z_{n-2} \\ z_{n-1} \\ z_n \end{Bmatrix}$$

and solve the following related problem instead of the problem in Eqs. (vi) :

$$\text{Minimize } \{\beta d^T \ 0 \ 0 \ M\} z$$

subject to

$$\begin{bmatrix} [\alpha] & 0 & -\frac{n}{\beta}b & \left(\frac{n}{\beta}b - [\alpha]\right)e \\ 0 & 0 & n & 0 \end{bmatrix} z = \begin{Bmatrix} 0 \\ 1 \end{Bmatrix}$$

$$e^T\bar{z} + z_{n-2} + z_{n-1} + z_n = 1 \qquad ...(vii)$$

$$z \geq 0$$

where e is an (m–1) component vector whose elements are all equal to 1,

1, z_{n-2} is a slack variable that absorbs the difference between 1 and the sum of other variables, z_{n-1} is constrained to have a value of 1/n, and M is given a large value (corresponding to the artificial variable z_n) to force z_n to zero when the problem stated in Eq. (iii) has a feasible solution. Equations (iv) are developed such that z is a solution to these equations, $X = \beta\bar{z}$ will be a solution to Eq. (iii) if Eq. (iii) have a feasible solution. Also, it can be verified that the interior point $z = (1/n)\ e$ is a feasible solution to Eq. (vii). Equation (vii) can be seen to be the desired from of Eq. (iii) except for a 1 on the right-hand side. This can be eliminated by subtracting the last constraint from the next-to-last constraint, to obtain the required form

$$\text{Minimize } \{\beta d^T \ 0 \ 0 \ M\} z$$

subject to

$$\begin{bmatrix} [\alpha] & 0 & -\frac{n}{\beta}b & \left(\frac{n}{\beta}b - [\alpha]e\right) \\ -e^T & -1 & (n-1) & -1 \end{bmatrix} z = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix}$$

$$e^T\bar{z} + z_{n-2} + z_{n-1} + z_n = 1 \qquad ...(viii)$$

$$z \geq 0$$

Note : When Eq. (viii) is solved, if the value of the artificial variable $z_n > 0,$ the original problem in Eqs. (iii) is infeasible. On the other hand, if the value of the slack variable $z_{n-2} = 0,$ the solution of the problem given by Eq. (iii) is unbounded.

Example-1 : Transform the following LP problem into a form required by Karmakar's method :

$$\text{Minimize } 2x_1 + 3x_2$$

subject to

$$3x_1 + x_2 - 2x_3 = 3$$

$$5x_1 - 2x_2 = 2$$

$$x_i \geq 0,\ i = 1,\ 2,\ 3$$

Solution : It can be seen that $d = \{2\ 3\ 0\}^T$, $[\alpha] = \begin{bmatrix} 3 & 1 & -2 \\ 5 & -2 & 0 \end{bmatrix}$, $b = \begin{Bmatrix} 3 \\ 2 \end{Bmatrix}$ and $X = \{x_1\ x_2\ x_3\}^T$. We define the integers m and n as $n = 6$ and $m = 3$ and choose $\beta = 10$ so that

$$\bar{z} = \frac{1}{10}\begin{Bmatrix} z_1 \\ z_2 \\ z_3 \end{Bmatrix}$$

Noting that $e = \{1\ 1\ 1\}^T$, Eq. (viii) can be expressed as

$$Minimize\ \{20\quad 30\quad 0\quad 0\quad 0\quad M\}\,z$$

subject to

$$\left[\begin{bmatrix} 3 & 1 & -2 \\ 5 & -2 & 0 \end{bmatrix}\begin{Bmatrix} 0 \\ 0 \end{Bmatrix} - \frac{6}{10}\begin{Bmatrix} 3 \\ 2 \end{Bmatrix} \cdot \left(\frac{6}{10}\begin{Bmatrix} 3 \\ 2 \end{Bmatrix} - \begin{bmatrix} 3 & 1 & -2 \\ 5 & -2 & 0 \end{bmatrix}\begin{Bmatrix} 1 \\ 1 \\ 1 \end{Bmatrix}\right)\right] z = 0$$

$$\{-\{1\quad 1\quad 1\} - 1\quad 5\quad -1\}\,z = 0$$

$$z_1 + z_2 + z_3 + z_4 + z_5 + z_6 = 1$$

$$z = \{z_1\quad z_2\quad z_3\quad z_4\quad z_5\quad z_6\}^T \geq 0$$

where M is a very large number. These equations can be seen to be in the desired form.

3. Algorithm

Starting from an interior feasible point $X^{(1)}$, Karmakar's method finds a sequence of points $X^{(2)}, X^{(3)}, \ldots$ using the following iterative procedure :

1. Initialize the process. Being at the center of the simplex as the initial feasible point $X^{(1)} = \left\{\frac{1}{n}\ \frac{1}{n} \cdots \frac{1}{n}\right\}^T$.

Set the iteration number as $k = 1$.

2. Test for optimality. Since $f = 0$ at the optimum point, we stop the procedure if the following convergence criterior is satisfied :

$$\left\| c^T X^{(k)} \right\| \leq \in \qquad (ix)$$

where $\in$ is a small number. If Eq. (viii) is not satisfied, go to step.

3. Compute the next point, $X^{(k+1)}$. For this, we first find a point $Y^{(k+1)}$ in the transformed unig simplex as

$$Y^{(k+1)} = \left\{\frac{1}{n}\ \frac{1}{n} \cdots \frac{1}{n}\right\}^T - \frac{\alpha\left([I] - [P]^T\left([P][P]^T\right)^{-1}[P]\right)\left[D\left(X^{(k)}\right)\right]c}{\|c\|\sqrt{n(n-1)}} \quad \ldots(x)$$

where $\|c\|$ is the length of the vector c, $[I]$ the identity matrix of order n, $\left[D\left(X^k\right)\right]$ an $n \times n$ matrix with all off-diagonal entries equal to 0, and diagonal entries are equal to the components of the vector $X^{(k)}$ as

$$\left[D(X)^{(k)}\right]_{ii} = x_i^{(k)},\ i = 1,\ 2,\ ....,\ n \qquad ...(xi)$$

$[P]$ is an $(m+1)\times n$ matrix whose first m rows are given by $[a]\left[D\left(X^{(k)}\right)\right]$ and the last row is composed of 1' is :

$$[P] = \begin{bmatrix} [\alpha]\left[D\left(X^{(k)}\right)\right] \\ 1 \quad\quad 1 \ .... \ 1 \end{bmatrix} \qquad ...(xii)$$

and the value of the parameter α is usually chosen as $\alpha = \frac{1}{4}$ to ensure convergence. Once $Y^{(k+1)}$ is found, the components of the new point $X^{(k+1)}$ are determined as

$$x_i^{(k+1)} = \frac{x_i^{(k)} y_i^{(k+1)}}{\sum_{r=1}^{n} x_r^{(k)} y_r^{(k+1)}},\ i = 1,\ 2,\ ....,\ n \qquad ...(xiii)$$

Set the new iteration number as $k = k+1$ and go to step 2.