M.S c Mathematics –SEM 3 Functional Analysis-L-9 CC-11 Unit 2

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Theorem based on linear transformation

Let N and N' be normed linear space and let $T: A \to N'$ be any linear transformation .If N is a finite dimensional,then T is continuous (or bounded) .

Proof

Let dim N=n

and $\{e_1, e_2, \ldots\ldots e_n\}$ be a basis for N .

Then to each $x \in N$,there exist unique scalars

$\alpha_1, \alpha_2, \alpha_3 \ldots \ldots\ldots \alpha_n$ such that

$$x = \sum_{i=1}^{n} \alpha_i e_i$$

Since T is linear, we have

$$T(x) = \sum_{i=1}^{n} \alpha_i \, T(e_i) \ldots \ldots (1)$$

We know that the suffices to consider zeroth norm on N defined by

$$||x_o|| = max|\alpha_i| \ldots \ldots (2)$$

If $|| \quad ||$ **is the norm on** N' **then (1)**

gives $||T(x)|| = ||\sum_{i=1}^{n} \alpha_i e_i|| \le \sum_{i=1}^{n} |\alpha_i|||e_i||$

$$\le \; ||x||_0| \sum_{i=1}^{n} ||e_i\,|. \ldots . (3)$$

since the basis is fixed

$\sum_{i=1}^{n} ||e_i||$ **is a positive constant.**

So , $M = \sum_{i=1}^{n} ||e_i\,||$

We have $||T(x)|| \le M \; ||x||_o$

Thus T is bounded and it is continuous.

Continuous Linear Transformation

Let E and F be normed linear spaces over the same scalar field K(which is either the real field R for both E & F or the complex field C for both E & F) and let T be a linear transformation of E into F.

We say that T is continuous iff $x_n \to x$ **in** E

$\Rightarrow T(x_n) \to T(x)$ **in F**

Note : Continuous linear transformation is a continuous linear operator.

Identity Transformation

For any normed linear space E, the identity transformation $I: E \to E$ **defined by** $Ix = x$ **for every** $x \in E$ **is a continuous linear transformation.**

i.e $x_n \to x$ **in E**

$\Rightarrow Ix_n \to I_x$ in E.

Zero Transformation

For **normed linear space E & F ,the Zero transformation**

$\boldsymbol{O}: \boldsymbol{E} \rightarrow \boldsymbol{F}$ **defined by** $\boldsymbol{O}(\boldsymbol{x}) = \boldsymbol{o}$ **for every** $\boldsymbol{x} \in \boldsymbol{E}$ **(where the R.H.S 0 is the zero element of F) is a continuous linear transformation**

In fact

$$\boldsymbol{x_n \rightarrow x \; in \; E}$$

$$\Rightarrow \boldsymbol{0x_n = 0 \rightarrow 0(x) = 0}$$