M.S c Mathematics –SEM 3 Functional Analysis-L-7 CC-11 Unit 1

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Equivalent Norms

Let a Linear space L be made into a Normed linear space in two ways, and let the two norms of a vector x in L be denoted by $||x||_1$ and $||x||_2$.

Then these norms are sai to be equivalent i.e $||\ \ ||_1 \sim ||\ \ ||_2$ if they generate the same topology on L.

Theorem

Let N be a normed linear space and suppose two norms e $||\ \ ||_1 and ||\ \ ||_2$ are defined on N .Then there norms are equivalent if and only if there exists positive real numbers m and m such that $m||x||_1 \leq ||x||_2 \leq M||x||_1$ for every x in N.

Proof

Let N_i be the NLS with the norm $||\ \ ||_i$(i=1,2)

Let $T(x) = x$

And we consider T as a linear transformation with domain N_1 and range N_2

Then $T^{-1}$is a linear transformation with domain N_1 and range N_2 such that

$$T(x) = x$$

$$\Leftrightarrow T^{-1}(x) = x$$

Now T is continuous

⇔T is bounded

⇔ ∃ positive number M such that

$$||T(X)||_2 \leq M||x||_1 \forall\ x \in N$$

$||x||_2 \leq M||x||_1 \forall x \in N$......................(1)

$$T(x) = x$$

Again $T^{-1}$is continuous

$\Leftrightarrow T^{-1}$is bounded

$\Leftrightarrow$ $\exists$ positive number K such that

$$||T^{-1}(x)||_1 \le K||x||_2 \forall x \in N_2$$

$$||x||_1 \le K||x||_2 [T^{-1}(x) = x]$$

$$\frac{1}{K}||x||_1 \le ||x||_2 [K > 0]$$

$m||x||_1 \le ||x||_2$....................(2)

Also T and $T^{-1}$are continuous

$\Leftrightarrow$Inverse images of open seta in $N_2 and N_1$ under T and T^{-1} respectively are open in $N_1$and N_2

$\Leftrightarrow$Open sets in $N_1$and $N_2$are the same.

[T and T^{-1} are identically transformation]

$\Leftrightarrow$ $||\ \ ||_1$and $||\ \ ||_2$ induce the same topology on N

So from (1) and (2)

We conclude that

$||\ \ ||_1$and $||\ \ ||_2$ are equivalent.

⇔There exist positive number m and M such that

$$\boldsymbol{m||x||_1 \leq ||x||_2 \leq M||x||_1 \; \forall x \in N}$$