e-content(lecture-8)

by

DR. ABHAY KUMAR (Guest Faculty)

P.G.Department of Mathematics

Patna University Patna

SEM-3 CC-11 UNIT-4(Functional Analysis)

Topic: Theorem on the Hilbert Space.

Theorem*: Let M be a closed linear subspace of a Hilbert space H and let x be a vector in H-M and let d be the distance from x to M*

i.e d=$inf\{\|x-z\|: z \in M\}$.

Then there exists a unique vector y_0 in M such that $\|x-y_0\| = d$ and y_0 is the unique vector of M for which $x-y_0$ is orthogonal to M.

Proof: *Let $x \in H$ and d=*$inf\{\|x-z\|: z \in M\}$. *then for each positive integer n,there exists $z_n \in M$ such that*

$d \leq \|x-z_n\| < d+\frac{1}{n}$ *so the sequence* $\|x-z_n\| \rightarrow d$

Since M is a subspace of H so it is convex set in H hence

$\frac{z_n+z_m}{2} \in M$ *so* $d \leq \left\| x - \frac{z_n+z_m}{2} \right\|$

Now by parallelogram law for $z_n - x\ , z_m - x$ *we have*

$$\|(z_n - x) + (z_m - x)\|^2 + \|(z_n - x) - (z_m - x)\|^2$$
$$= 2\|z_n - x\|^2 + 2\|z_m - x\|^2$$
$$\Rightarrow \|z_n - z_m\|^2 = 2\|z_n - x\|^2 + 2\|z_m - x\|^2 - \|(z_n + z_m) - 2x\|^2$$
$$\leq 2d^2 + 2d^2 - 4d^2 = 0 \quad as\ m,n \to \infty.$$

So (z_n) *is a Cauchy sequence in M and since M is closed in complete space H so it is complete .Hence there exists* y_0 *in M such that* $z_n \to y_0$*.Hence*

$x - z_n \to x - y_0 \Rightarrow \|x - z_n\| \to \|x - y_0\|$.

$\|x - y_0\| = \lim_{n\to\infty} \|x - z_n\| = d$ *thus there exists*

y_0 *in M such that* $\|x - y_0\| = d$.

We have to show that y_0 *is the unique vector of M.*

Let y_1 *is a vector of M such that* $\|x - y_1\| = d$.

Then $\frac{y_0 + y_1}{2} \in M$. *And hence* $\left\| x - \frac{y_0 + y_1}{2} \right\| \geq d$.

By parallelogram Law for $y_0 - x, y_1 - x$ *we have*

$$\|(y_0 - x) + (y_1 - x)\|^2 + \|(y_0 - x) - (y_1 - x)\|^2$$

$$= 2\|y_0 - x\|^2 + 2\|y_1 - x\|^2$$

$$\|y_0 - y_1\|^2 \leq 2\|y_0 - x\|^2 + 2\|y_1 - x\|^2 - 4d^2$$

$$\leq 2d^2 + 2d^2 - 4d^2 = 0$$

$$\|y_0 - y_1\|^2 \leq 0$$

Thus $\|y_0 - y_1\| \leq 0$. *but* $\|y_0 - y_1\| \geq 0$ *so we have*

$\|y_0 - y_1\| = 0$ hence $y_0 - y_1 = 0 \Rightarrow y_0 = y_1$

Hence there exists a unique vector y_0 *in M*

such that $\|x - y_0\| = d$.

We have to show that $x - y_0$ *is orthogonal to M.*

Let $z \in M$ *with* $\|z\| = 1$ *then*

$w = y_0 + (x - y_0, z)z \in M$ and we have

$$\|x - y_0\|^2 \leq \|x - w\|^2 = (x - w, x - w)$$

$$= \|x - y_0\|^2 - |(x - y_0, z)|^2$$

$$\Rightarrow |(x - y_0, z)| = 0 \Rightarrow (x - y_0, z) = 0$$

$$\Rightarrow x - y_0 \perp z$$

Hence $x - y_0$ *is orthogonal to M.*

Conversely

let $y_0 \in M$ *and* $x - y_0$ *is orthogonal to M.*

then for any $z \in M$ we have $y_0 - z \in M$

So that $x - y_0 \perp y_0 - z$

Hence by pythagorian theorem

$$\|x - z\|^2 = \|(x - y_0) + (y_0 - z)\|^2$$

$$= \|(x - y_0)\|^2 + \|(y_0 - z)\|^2$$

Thus $\| x - y_0\| \leq \|x - z\|$ $if\, y_0 \neq z$ *this show that* y_0 *is unique in M.*

END.