

INTEGER PROGRAMMING PROBLEM (M.Sc. Sem-III)

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INTEGER PROGRAMMING PROBLEM

1. Cutting-Plane Algorithm

Example-1

Consider the following ILP.

$$\text{Maximize } z = 7x_1 + 10x_2$$

subject to

$$-x_1 + 3x_2 \leq 6$$

$$7x_1 + x_2 \leq 345$$

$$x_1, x_2 \geq 0 \text{ and integer}$$

The cutting-plane algorithm modifies the solution space by adding *cuts* that produce an optimum integer extreme point.

Initially, we start with the continuous LP optimum $z = 66\frac{1}{2}$, $x_1 = 4\frac{1}{2}$, $x_2 = 3\frac{1}{2}$. Next, we add cut I, which produces the (continuous) LP optimum solution $z = 62$, $x_1 = 4\frac{4}{7}$, $x_2 = 3$. Then, we add cut II, which together with cut I and the original constraints, produces the LP optimum $z = 58$, $x_1 = 4$, $x_2 = 3$. The last solution is all integer, as desired.

The added cuts do not eliminate any of the original feasible integer points, but must pass through at least one feasible or infeasible integer point. These are basic requirements of any cut.

It is purely accidental that a 2-variable problem used exactly 2 cuts to reach the optimum integer solution. In general, the number of cuts, though finite, is independent of the size of the problem, in the sense that a problem with a small number of variables and constraints may require more cuts than a larger problem.

Next, we use the same example to show how the cuts are constructed and implemented algebraically.

Given the slacks x_3 and x_4 for constraints 1 and 2, the optimum LP tableau is given as

Basic	x_1	x_2	x_3	x_4	Solution
z	0	0	$\frac{63}{22}$	$\frac{31}{22}$	$66\frac{1}{2}$
x_2	0	1	$\frac{7}{22}$	$\frac{1}{22}$	$3\frac{1}{2}$
x_1	1	0	$-\frac{1}{22}$	$\frac{3}{22}$	$4\frac{1}{2}$

The optimum continuous solution is $z = 66\frac{1}{2}$, $x_1 = 4\frac{1}{2}$, $x_2 = 3\frac{1}{2}$, $x_3 = 0$, $x_4 = 0$. The cut is developed under the assumption that all the variables (including the slacks x_3 and x_4) are integer. Note also that because all the original objective coefficients are integer in this example, the value of z is integer as well.

The information in the optimum tableau can be written explicitly as :

$$z + \frac{63}{22}x_3 + \frac{31}{22}x_4 = 66\frac{1}{2} \quad (\text{z-equation})$$

$$x_2 + \frac{7}{22}x_3 + \frac{1}{22}x_4 = 3\frac{1}{2} \quad (x_2 - \text{equation})$$

$$x_1 - \frac{1}{22}x_3 + \frac{3}{22}x_4 = 4\frac{1}{2} \quad (x_1 - \text{equation})$$

A constraint equation can be used as a **source row** for generating a cut, provided its right-hand side is fractional. We also note that the z-equation can be used as a source row because z happens to be integer in this example. We will demonstrate how a cut is generated from each of these source rows, starting with the z-equation.

First, we factor out all the non-integer coefficients of the equation into an integer value and a fractional component, provided that the resulting fractional component is strictly positive. For example,

$$\frac{5}{2} = \left(2 + \frac{1}{2}\right)$$

$$-\frac{7}{3} = \left(-3 + \frac{2}{3}\right)$$

The factoring of the z-equation yields

$$z + \left(2 + \frac{19}{22}\right)x_3 + \left(1 + \frac{9}{22}\right)x_4 = \left(66 + \frac{1}{2}\right)$$

Moving all the integer components to the left-hand side and all the fractional components to the right-hand side, we get

$$z + 2x_3 + 1x_4 - 66 = -\frac{19}{22}x_3 - \frac{9}{22}x_4 + \frac{1}{2} \quad \dots(1)$$

Because x_3 and x_4 are non-negative and all fractions are originally strictly positive, the right-hand side must satisfy the following inequality :

$$-\frac{19}{22}x_3 - \frac{9}{22}x_4 + \frac{1}{2} \leq \frac{1}{2} \quad \dots(2)$$

Next, because the left-hand side in Equation (1), $z + 2x_3 + 1x_4 - 66$, is an integer value by construction, the right-hand side, $-\frac{19}{22}x_3 - \frac{9}{22}x_4 + \frac{1}{2}$, must also be integer. It

then follows that (2) can be replaced with the inequality :

$$-\frac{19}{22}x_3 - \frac{9}{22}x_4 + \frac{1}{2} \leq 0$$

This result is justified because an integer value $\leq \frac{1}{2}$ must necessarily be ≤ 0

The last inequality is the desired cut and it represents a necessary (but not sufficient) condition for obtaining an integer solution. It is also referred to as the **fractional cut** because all its coefficient are fractions.

Because $x_3 = x_4 = 0$ in the optimum continuous LP tableau given above, the current continuous solution violates the cut (because it yields $\frac{1}{2} \leq 0$). Thus, if we add this cut to the optimum tableau, the resulting optimum extreme point moves the solution toward satisfying the integer requirements.

Before showing how a cut is implemented in the optimal tableau, we will demonstrate how cuts can also be constructed from the constraint equations. Consider the x_1 - row;

$$x_1 - \frac{1}{22}x_3 + \frac{3}{22}x_4 = 4\frac{1}{2}$$

Factoring the equation yields

$$x_1 + \left(-1 + \frac{21}{22}\right)x_3 + \left(0 + \frac{3}{22}\right)x_4 = \left(4 + \frac{1}{2}\right)$$

The associated cut is

$$-\frac{21}{22}x_2 - \frac{3}{22}x_4 + \frac{1}{2} \leq 0$$

Similarly, the x_2 – equation

$$x_2 + \frac{7}{22}x_3 + \frac{1}{22}x_4 = 3\frac{1}{2}$$

is factored as

$$x_2 + \left(0 + \frac{7}{22}\right)x_3 + \left(0 + \frac{1}{22}\right)x_4 = 3 + \frac{1}{2}$$

Hence, the associated cut is given as

$$-\frac{7}{22}x_3 - \frac{1}{22}x_4 + \frac{1}{2} \leq 0$$

Any one of three cuts given above can be used in the first iteration of the cutting-plane algorithm. It is not necessary to generate all three cuts before selecting one.

Arbitrarily selecting the cut generated from the x_2 – row, we can write it in equation form as

$$-\frac{7}{22}x_3 - \frac{1}{22}x_4 + s_1 = -\frac{1}{2}, s_1 \geq 0 \quad (\text{Cut I})$$

This constraint is added to the LP optimum tableau as follows :

Basics	x_1	x_2	x_3	x_4	s_1	Solution
z	0	0	$\frac{63}{22}$	$\frac{31}{22}$	0	$66\frac{1}{2}$
x_2	0	1	$\frac{7}{22}$	$\frac{1}{22}$	0	$3\frac{1}{2}$
x_1	1	0	$-\frac{1}{22}$	$\frac{3}{22}$	0	$4\frac{1}{2}$
s_1	0	0	$-\frac{7}{22}$	$-\frac{1}{22}$	1	$-\frac{1}{2}$

The tableau is optimal but infeasible. We apply the dual simplex method to recover feasibility, which yields

Basic	x_1	x_2	x_3	x_4	s_1	Solution
z	0	0	0	1	9	62
x_2	0	1	0	0	1	3
x_1	1	0	0	$\frac{1}{7}$	$-\frac{1}{7}$	$4\frac{1}{7}$
x_3	0	0	1	$\frac{1}{7}$	$-\frac{22}{7}$	$1\frac{4}{7}$

The last solution is still non-integer in x_1 and x_3 . Let us arbitrarily select x_1 as the next source row-that is,

$$x_1 + \left(0 + \frac{1}{7}\right)x_4 + \left(-1 + \frac{6}{7}\right)s_1 = 4 + \frac{4}{7}$$

The associated cut is

$$-\frac{1}{7}x_4 - \frac{6}{7}s_1 + s_2 = -\frac{4}{7}, s_2 \geq 0 \text{ (Cut II)}$$

Basics	x_1	x_2	x_3	x_4	s_1	s_2	Solution
z	0	0	0	1	9	0	62
x_2	0	1	0	0	1	0	3
x_1	1	0	0	$\frac{1}{7}$	$-\frac{1}{7}$	0	$4\frac{4}{7}$
x_3	0	0	1	$\frac{1}{7}$	$-\frac{22}{7}$	0	$1\frac{4}{7}$
s_2	0	0	0	$-\frac{1}{7}$	$-\frac{6}{7}$	1	$-\frac{4}{7}$

The dual simplex method yields the following tableau :

Basic	x_1	x_2	x_3	x_4	s_1	s_2	Solution
z	0	0	0	0	3	7	58
x_2	0	1	0	0	1	0	3
x_1	1	0	0	0	-1	1	4
x_3	0	0	1	0	-4	1	1
x_4	0	0	0	1	6	-7	4

The optimum solution ($x_1 = 4, x_2 = 3, z = 58$) is all integer. It is not accidental that all the coefficients of the last tableau are integers, a property of the implementation of the fractional cut.