E-content(lecture-7)

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SEM-3 CC-11 UNIT-4(Functional analysis)

Topic: *Theorem on Hilbert space*

Theorem:

(Lemma of F.Riesz on closed convex set in a Hilbert space)

Let A be a closed convex set in a Hilbert space H .Then there exists a unique vector $x_0 \in A$ such that

$\| x_0\| \leq \| y\|$ *for every* $y \in A$ *.*

In other words, a closed convex subset in a Hilbert space H contains a unique vevtor of smallest norm.

Proof: Let $\beta = glb\{\| x\|: \ x \in A\}$. *Then for each positive integer* n *,there exists* $x_n \in A$ *such that*

$\beta \leq \| x_n\| \leq \beta + \frac{1}{n}$ *Hence the sequence* (x_n)*in* A

Is such that $\| x_n \| \to \beta$. *since A is convex so* $\frac{x_m+x_n}{2} \in A$

and hence $\| \frac{x_m+x_n}{2} \| \geq \beta$ *by the parallelogram law*

we get

$$\|x_m - x_n\|^2 = 2\|x_m\|^2 + 2\|x_n\|^2 - \|x_m + x_n\|^2$$

$$\leq 2\|x_m\|^2 + 2\|x_n\|^2 - 4\beta^2$$

$$\to 2\beta^2 + 2\beta^2 - 4\beta^2 = 0 \; as \; m, n \to \infty.$$

Hence $\|x_m - x_n\| \to 0 \quad as \; m, n \to \infty$.

So (x_n) *is a Cauchy sequence in A.Since A is closed in*

the complete metric space H. Hence A is also complete

so there exists $x_0 \in A$ *such that* $x_n \to x_0$. *since the*

norm is continuous function so $\|x_n\| \to \|x_0\|$ *.*

But $\| x_n \| \to \beta$ *Since the sequence has the unique limit*

Hence $\|x_0\| = \beta$. *Thus* x_0*is a vector in A with smallest*

norm .

We have to show that x_0*is a unique vector in A.*

Let x_1*is an another vector in A different from* x_0

Such that $\|x_1\| = \beta$ .Then $\frac{x_0+x_1}{2} \in A$

and hence $\|\frac{x_0+x_1}{2}\| \geq \beta$.

But by the parallelogram law we have

$$\|\frac{x_0+x_1}{2}\|^2 = 2\|\frac{x_0}{2}\|^2 + 2\|\frac{x_1}{2}\|^2 - \|\frac{x_0-x_1}{2}\|^2$$

$$< \frac{1}{2}\|x_0\|^2 + \frac{1}{2}\|x_1\|^2 \text{ (since } \|x_0 - x_1\| > 0)$$

$$= \frac{1}{2}\beta^2 + \frac{1}{2}\beta^2 = \beta^2.$$

hence $\|\frac{x_0+x_1}{2}\| < \beta$.

Which is contradiction because $\|\frac{x_0+x_1}{2}\| \geq \beta$.

Hence x_0 is a unique vector in A with smallest norm.

Theorem : Let M be a closed convex subset of a normed linear space H and let x be a vector in H .Then x +M is also a closed convex subset in H.

Proof:Let A= x +M={ $x + m : m \in$M}

) Let $y_1, y_2 \in A$ and $0 \leq \alpha \leq 1$.Then

$y_1 = x + m_1, y_2 = x + m_2$ for some m_1, $m_2 \in$ M.

$\alpha y_1 + (1-\alpha) y_2 = \alpha (x + m_1) + (1-\alpha)(x + m_2)$

$= x + (\alpha m_1 + (1-\alpha) m_2) \in x + M$

Since $(\alpha m_1 + (1-\alpha) m_2) \in M$ *and M is convex set*

So $x + M$ *is also a convex set since M is closed set and so* $x + M$ *is a closed set thus A is closed and convex set in H.*

END.