

# **e-content**

**by**

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**SEMESTER – IV**

**Elective Paper (Mathematical Methods)**

**Topic : Bessel's Equation**

## Topic : Bessel's Equation

The differential equation of the form  $\frac{d^2y}{dx^2} + \frac{1}{x} \frac{dy}{dx} + y = 0$  .....  
(i) is called the Bessel's equation for  $n = 0$

### Solution of the diff. equation

$$\frac{d^2y}{dx^2} + \frac{1}{x} \frac{dy}{dx} + y = 0 \quad \dots \dots \dots \text{(i)}$$

**Solution :** Suppose that

$$y = \sum_{r=0}^{\infty} a_r x^{k+r} \quad \begin{array}{l} \text{be the solution of (i)} \\ \text{where } a_0 \neq 0 \end{array}$$

$$\therefore \frac{dy}{dx} = \sum_{r=0}^{\infty} a_r (k+r)x^{k+r-1}$$

$$\text{and } \frac{d^2y}{dx^2} = \sum_{r=0}^{\infty} a_r (k+r)(k+r-1)x^{k+r-2}$$

Putting these values in (i), we get

$$\sum_{r=0}^{\infty} a_r \left[ (k+r)(k+r-1)x^{k+r-2} + \frac{1}{x} (k+r)x^{k+r-1} + x^{k+r} \right] = 0$$

$$\text{or } \sum_{r=0}^{\infty} a_r [(k+r)^2 x^{k+r-2} + x^{k+r}] = 0 \quad \dots \dots \dots \text{(2)}$$

Equating the coefficient of  $x^{k-2}$ , we get

$$a_0 K^2 = 0$$

(3)

Again, equating the coefficient of  $x^{k-1}$

We have

$$\Rightarrow a_1 = 0$$

Again, equating the coefficient of  $x^{k+r}$

We have

$$\therefore a_{r+2} = -\frac{a_r}{(k+r+2)^2}$$

Putting  $r = 1, 3, 5, \dots$  etc.

We get  $a_1 = 0, a_3 = 0, a_5 = 0, \dots \dots \dots$

Again, putting  $r = 0, 2, 4, \dots \dots \dots$  etc.

$$a_2 = -\frac{a_0}{2^2}$$

$$a_4 = -\frac{a_2}{4^2} = -\frac{a_0}{2^2 \cdot 4^2} \dots \dots \dots$$

Since  $y = \sum_{r=0}^{\infty} a_r x^r \quad (\because k = 0)$

$$\therefore y = a_0 \left( 1 - \frac{x^2}{2^2} + \frac{x^4}{2^2 \cdot 4^2} - \frac{x^6}{2^2 \cdot 4^2 \cdot 6^2} \dots \dots \dots \right)$$

If  $a_6 = 1$  Then, this solution is denoted by  $J_0(x)$

$$\text{Thus } J_0(x) = \left( 1 - \frac{x^2}{2^2} + \frac{x^4}{2^2 \cdot 4^2} - \frac{x^6}{2^2 \cdot 4^2 \cdot 6^2} \dots \dots \dots \right)$$

Where  $J_0(x)$  is called Bessel's function of Zeroeth order.

### Recurrence formulae $J_n(x)$

**Prove that :- (I)**  $x \cdot J_n'(x) = n \cdot J_n(x) - x \cdot J_{n+1}(x)$

**Proof :** We know that

$$J_n(x) = \sum_{r=0}^{\infty} (-1)^r \left( \frac{x}{2} \right)^{n+2r} \cdot \frac{1}{[r] (n+r+1)}$$

where  $n$  is a positive integer.

Differentiating w.r. to  $x$ , we get

$$J_n(x) = \sum_{r=0}^{\infty} (-1)^r \cdot \frac{n+2r}{\lfloor r \rfloor (n+r+1)} \cdot \frac{1}{2} \left( \frac{x}{2} \right)^{n+2r-1}$$

$$x \cdot J_n(x) = \sum_{r=0}^{\infty} (-1)^r \cdot \frac{(n+2r)}{\lfloor r \rfloor (n+r+1)} \cdot \left( \frac{x}{2} \right)^{n+2r}$$

$$= \sum_{r=0}^{\infty} (-1)^r \cdot \frac{n}{\lfloor r \rfloor (n+r+1)} \cdot \left( \frac{x}{2} \right)^{n+2r}$$

$$+ \sum_{r=0}^{\infty} (-1)^r \cdot \frac{2r}{\lfloor r \rfloor (n+r+1)} \cdot \left( \frac{x}{2} \right) \left( \frac{x}{2} \right)^{n+2r-1}$$

$$= n J_n(x) + x \sum_{r=1}^{\infty} (-1)^r \cdot \frac{1}{\lfloor r-1 \rfloor (n+r+1)} \cdot \left( \frac{x}{2} \right)^{n+2r-1}$$

$$= n J_n(x) - x \sum_{r=1}^{\infty} (-1)^{r-1} \cdot \frac{1}{\lfloor r-1 \rfloor (n+r+1)} \cdot \left( \frac{x}{2} \right)^{n+2r-1}$$

$$= n J_n(x) - x \sum_{s=0}^{\infty} (-1)^s \cdot \frac{1}{\lfloor s \rfloor (n+1+s+1)} \cdot \left( \frac{x}{2} \right)^{(n+1)+2s} \quad \text{where } r-1=s$$

$$= n J_n(x) - x J_{n+1}(x)$$

$$\text{Hence, } x J'_n(x) = n J_n(x) - x J_{n+1}(x)$$

**Prove that :-** (II)  $x J_n'(x) = n J_n(x) + x J_{n-1}(x)$

**Proof :** We know that

$$J_n(x) = \sum_{r=0}^{\infty} (-1)^r \left( \frac{x}{2} \right)^{n+2r} \cdot \frac{1}{[r]_r (n+r+1)}$$

Diff. w. r. to  $x$  we get

$$= J_n'(x) = \sum_{r=0}^{\infty} (-1)^r \cdot \frac{(n+2r)}{[r]_r (n+r+1)} \frac{1}{2} \cdot \left( \frac{x}{2} \right)^{n+2r-1}$$

$$\therefore x J_n'(x) = \sum_{r=0}^{\infty} (-1)^r \cdot \frac{n+2r}{[r]_r (n+r+1)} \cdot \left( \frac{x}{2} \right)^{n+2r}$$

$$= \sum_{r=0}^{\infty} (-1)^r \cdot \frac{(2n+2r-n)}{[r]_r (n+r+1)} \cdot \left( \frac{x}{2} \right)^{n+2r}$$

$$= -n \sum_{r=0}^{\infty} (-1)^r \cdot \frac{1}{[r]_r (n+r+1)} \cdot \left( \frac{x}{2} \right)^{n+2r}$$

$$+ \sum_{r=0}^{\infty} (-1)^r \cdot \frac{2n+2r}{[r]_r (n+r+1)} \cdot \left( \frac{x}{2} \right)^{n+2r}$$

$$= -n J_n(x) + \sum_{r=0}^{\infty} (-1)^r \cdot \frac{2(n+r)}{[r]_r (n+r+1)} \cdot \left( \frac{x}{2} \right)^{n+2r-1} \cdot \left( \frac{x}{2} \right)$$

$$\begin{aligned}
&= -n J_n(x) + x \sum_{r=0}^{\infty} (-1)^r \cdot \frac{1}{[r]_x (n+r)} \cdot \left( \frac{x}{2} \right)^{n+2r-1} \\
&= -n J_n(x) + x \sum (-1)^r \cdot \frac{1}{[r]_x (n-1+r+1)} \cdot \left( \frac{x}{2} \right)^{(n-1)+2r} \\
&\quad = n J_n(x) + x J_{n-1}(x)
\end{aligned}$$

Hence  $x J'_n(x) = -n J_n(x) + x J_{n-1}(x)$

**Prove that :- (III)**  $2J'_n(x) = J_{n-1}(x) + J_{n+1}(x)$

**Proof :** We know that

$$J_n(x) = \sum_{r=0}^{\infty} (-1)^r \cdot \frac{1}{[r]_x (n+r+1)} \cdot \left( \frac{x}{2} \right)^{n+2r}$$

Diff w.r. to  $x$  we get

$$\begin{aligned}
&2J'_n(x) = \sum_{r=0}^{\infty} (-1)^r \cdot \frac{2(n+2r)}{[r]_x (n+r+1)} \cdot \frac{1}{2} \left( \frac{x}{2} \right)^{n+2r-1} \\
&= \sum_{r=0}^{\infty} (-1)^r \cdot \frac{n+r+r}{[r]_x (n+r+1)} \cdot \left( \frac{x}{2} \right)^{n+2r-1}
\end{aligned}$$

$$\begin{aligned}
&= \sum (-1)^r \cdot \frac{(n+r)}{\lfloor r \rfloor (n+r+1)} \left( \frac{x}{2} \right)^{n+2r-1} \\
&\quad + \sum_{r=0}^{\infty} (-1)^r \cdot \frac{r}{\lfloor r \rfloor (n+r+1)} \left( \frac{x}{2} \right)^{n+2r-1} \\
&= \sum_{r=0}^{\infty} (-1)^r \cdot \frac{1}{\lfloor r \rfloor (n+r)} \left( \frac{x}{2} \right)^{n+2r-1} \\
&\quad - \sum_{r=0}^{\infty} (-1)^{r-1} \cdot \frac{1}{\lfloor r-1 \rfloor (n+r+1)} \left( \frac{x}{2} \right)^{n+2r-1} \\
&= \sum_{r=0}^{\infty} (-1)^r \cdot \frac{1}{\lfloor r \rfloor (n-1+r+1)} \left( \frac{x}{2} \right)^{\overline{n-1}+2r} \\
&\quad - \sum (-1)^s \cdot \frac{1}{\lfloor s \rfloor (n+1+s+1)} \left( \frac{x}{2} \right)^{n+1+2s}
\end{aligned}$$

where  $r-1 = s$

$$= J_{n-1}(x) - J_{n+1}(x)$$

$$\text{Hence, } 2J_n'(x) = J_{n-1}(x) - J_{n+1}(x)$$

**Prove that :** (IV)  $2n J_n(x) = x [J_{n-1}(x) + J_{n+1}(x)]$

**Proof :** We know that

$$\begin{aligned}
 J_n(x) &= \sum_{r=0}^{\infty} (-1)^r \cdot \frac{1}{[r]_r (n+r+1)} \cdot \left( \frac{x}{2} \right)^{n+2r} \\
 \therefore 2n J_n(x) &= \sum_{r=0}^{\infty} (-1)^r \cdot \frac{(2n+2r-2r)}{[r]_r (n+r+1)} \cdot \left( \frac{x}{2} \right)^{n+2r} \\
 &= \sum_{r=0}^{\infty} (-1)^r \cdot \frac{2(n+r)}{[r]_r (n+r+1)} \cdot \left( \frac{x}{2} \right)^{n+2r} \\
 &\quad - \sum_{r=0}^{\infty} (-1)^r \cdot \frac{2r}{[r]_r (n+r+1)} \cdot \left( \frac{x}{2} \right)^{n+2r} \\
 &= x \sum_{r=0}^{\infty} (-1)^r \cdot \frac{1}{[r]_r (n-1+r-1)} \cdot \left( \frac{x}{2} \right)^{n-1+2r} \\
 &\quad + x \sum_{r=0}^{\infty} (-1)^{r-1} \cdot \frac{1}{[r-1]_{r-1} (n+r+1)} \cdot \left( \frac{x}{2} \right)^{n+2r-1} \\
 &= x J_{n-1}(x) + x \sum_{s=0}^{\infty} (-1)^s \cdot \frac{1}{[s]_s (n+1+s+1)} \cdot \left( \frac{x}{2} \right)^{n+2r-1}
 \end{aligned}$$

where  $r-1 = s$

$$= x J_{n-1}(x) = x J_{n+1}(x)$$

$$\text{Hence, } 2n J_n(x) = x[J_{n-1}(x) + J_{n+1}(x)]$$

**Prove that :- (V)**  $\frac{d}{dx}[x^{-n} \cdot J_n(x)] = -x^{-n} J_{n+1}(x)$

**Proof :** 
$$\begin{aligned} & \frac{d}{dx}[x^{-n} J_n(x)] \\ &= -n x^{-n-1} J_n(x) + x^{-n} J'_n(x) \\ &= x^{-n-1} [-n J_n(x) + x J'_n(x)] \\ &= x^{-n-1} [-x J_{n+1}(x)] \quad (\text{from Rcc. formula I}) \\ &= x^{-n} [J_{n+1}(x)] \end{aligned}$$

Hence,  $\frac{d}{dx}[x^{-n} J_n(x)] = -x^{-n} J_{n+1}(x)$

**Prove that:- (VI)**  $\frac{d}{dx}[x^n J_n(x)] = x^n J_{n-1}(x)$

**Proof :** 
$$\begin{aligned} & \frac{d}{dx}[x^n J_n(x)] = nx^{n-1} J_n(x) + x^n J'_n(x) \\ &= x^{n-1} [n J_n(x) + x^n J'_n(x)] \\ &= x^{n-1} \cdot x J_{n-1}(x) \quad (\text{from Rcc. formula II}) \end{aligned}$$

$$= x^n J_{n-1}(x)$$

$$\text{Hence } \frac{d}{dx} [x^n J_n(x)] = x^n J_{n-1}(x)$$

**Generating function for  $J_n(x)$**

**Prove that when  $n$  is a positive integer  $J_n(x)$  is the coefficient of  $z^n$  in the expansion of  $e^{x(z-\frac{1}{z})/2}$  in ascending and descending powers of  $z$ . Also prove that  $J_n(x)$  is the coefficient of  $z^{-n}$  multiplied by  $(-1)^n$  in the expansion of above expression.**

**Proof :** Since  $e^{x(z-\frac{1}{z})/2}$

$$= e^{\frac{xz}{2}} \cdot e^{\frac{-x}{2z}}$$

$$= \left[ 1 + \frac{xz}{2} + \frac{1}{2} \left( \frac{xz}{2} \right)^2 + \dots + \frac{1}{n} \left( \frac{xz}{2} \right)^n \right]$$

$$+ \frac{1}{n+1} \left( \frac{xz}{2} \right)^{n+1} + \frac{1}{n+2} \left( \frac{xz}{2} \right)^{n+2} + \dots \left] \right.$$

$$\bullet \left[ 1 - \frac{x}{2z} + \frac{1}{2} \left( \frac{x}{2z} \right)^2 + \dots + \frac{(-1)^n}{n} \left( \frac{x}{2z} \right)^n \right]$$

$$+ \frac{(-1)^{n+1}}{[n+1]} \left( \frac{x}{2z} \right)^{n+1} + \frac{(-1)^{n+2}}{[n+2]} \left( \frac{x}{2z} \right)^{n+2} + \dots$$

So, the coefficient of  $z^n$  in this product

$$= \frac{1}{[n]} \left( \frac{x}{2} \right)^n - \frac{1}{[n+1]} \left( \frac{x}{2} \right) \left( \frac{x}{2} \right)^{n+1}$$

$$+ \frac{1}{[n+2]} \cdot \frac{1}{[2]} \left( \frac{x}{2} \right)^2 \left( \frac{x}{2} \right)^{n+2} + \dots$$

$$= \frac{1}{[n]} \left( \frac{x}{2} \right)^n - \frac{1}{[n+1]} \left( \frac{x}{2} \right)^{n+2} + \frac{1}{[2][n+2]} \left( \frac{x}{2} \right)^{n+4} + \dots$$

$$= \frac{(-1)^0}{[1][n+1]} \left( \frac{x}{2} \right)^n + \frac{(-1)}{[1][n+2]} \left( \frac{x}{2} \right)^{n+2}$$

$$+ \frac{(-1)^2}{[2][n+3]} \left( \frac{x}{2} \right)^{n+4} + \dots$$

$$= \sum_{r=0}^n (-1)^r \cdot \frac{1}{\lceil r \rceil (n+r+1)} \cdot \left( \frac{x}{2} \right)^{n+2r}$$

$$= J_n(x)$$

Similarly, the coefficient of  $z^{-n}$  in this product.

$$= \frac{(-1)^n}{\lfloor n \rfloor} \left( \frac{x}{2} \right)^n + \frac{(-1)^{n+1}}{\lfloor n+1 \rfloor} \left( \frac{x}{2} \right) \cdot \left( \frac{x}{2} \right)^{n+1}$$

$$+ \frac{(-1)^{n+2}}{\lfloor n+2 \rfloor} \cdot \frac{1}{\lfloor 2 \rfloor} \left( \frac{x}{2} \right)^2 \cdot \left( \frac{x}{2} \right)^{n+2} + \dots$$

$$= (-1)^n \left[ \frac{1}{\lfloor n \rfloor} \left( \frac{x}{2} \right)^n + \frac{(-1)}{\lceil (n+2) \rceil} \left( \frac{x}{2} \right)^{n+2} \right]$$

$$+ \frac{(-1)^2}{\lfloor 2 \rfloor \lceil (n+3) \rceil} \left( \frac{x}{2} \right)^{n+4} + \dots$$

$$= (-1)^n J_n(x)$$

It proves.