M.S c Mathematics –SEM 3 Rigid Dynamics

CC-13 Unit 1

Topic- Obtain Lagrange's differential equation for holonomic dynamical system

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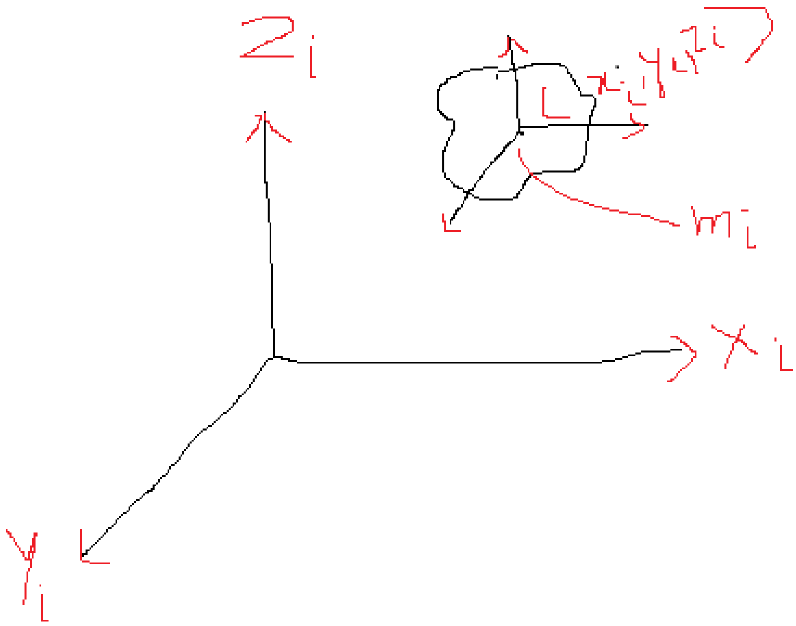
Obtain Lagrange's differential equation for holonomic dynamical system

Proof: Let m_i be the mass of the particle situated at (x_i, y_i, z_i) where external forces have component (X_i, Y_i, Z_i).

The components of reversed effective forces and external forces are

$$-m_i\ddot{x}_i + X_i, -m_i\ddot{y}_i + Y_i, -m_i\ddot{z}_i + Z_i$$



Let the system undergo a virtual displacement so that the changes in

$(\boldsymbol{x_i}, \boldsymbol{y_i}, \boldsymbol{z_i})$ **are** $(\boldsymbol{\partial x_i}, \boldsymbol{\partial y_i}, \boldsymbol{\partial z_i})$

The virtual workdone by reversal ffective forces and external forces at

$(\boldsymbol{x_i}, \boldsymbol{y_i}, \boldsymbol{z_i})$ **is**

$$(-\boldsymbol{m_i \ddot{x}_i} + \boldsymbol{X_i})\boldsymbol{\partial x_i} + (-\boldsymbol{m_i \ddot{y}_i} + \boldsymbol{Y_i})\boldsymbol{\partial y_i} \\ +(-\boldsymbol{m_i \ddot{z}_i} + \boldsymbol{Z_i})\boldsymbol{\partial z_i}$$

Hence, the total virtual work of the external forces and reversed effective force is

$$\sum\{(-\boldsymbol{m_i \ddot{x}_i} + \boldsymbol{X_i})\boldsymbol{\partial x_i} + (-\boldsymbol{m_i \ddot{y}_i} + \boldsymbol{Y_i})\boldsymbol{\partial y_i} \\ +(-\boldsymbol{m_i \ddot{z}_i} + \boldsymbol{Z_i})\boldsymbol{\partial z_i}\}$$

But ,the reversed effective forces and external forces constitute the system in equilibrium.

so ,the sum of their virtual work shall be zero

i.e

$$\sum_{i=1}^{n}\{(-m_i\ddot{x}_i + X_i)\partial x_i + (-m_i\ddot{y}_i + Y_i)\partial y_i$$
$$+(-m_i\ddot{z}_i + Z_i)\partial z_i\} = 0$$

or $\sum_{i=1}^{n} m_i\,(\ddot{x}_i\partial x_i + \ddot{y}_i\partial y_i + \ddot{z}_i\partial z_i)$

$= \sum_{i=1}^{n}(X_i\,\partial x_i + Y_i\partial y_i + Z_i\partial z_i$ **.....................(1)**

Now,

$$x_i = x_i(q_1, q_2, \dots\ ..q_n, t)$$
$$y_i = y_i(q_1, q_2, \dots\ ..q_n, t)$$

$$z_i = z_i(q_1, q_2, \dots\ ..q_n, t)$$

Then $\partial x_i = \sum_{r=1}^{n}\frac{\partial x_i}{\partial q_r}\partial q_r$

$$\partial y_i = \sum_{r=1}^{n} \frac{\partial y_i}{\partial q_r} \partial q_r$$

$\partial z_i = \sum_{r=1}^{n} \frac{\partial z_i}{\partial q_r} \partial q_r$

Now from R.H.S of equation (1),we have

$$\sum_{i=1}^{n} (X_i \partial x_i + Y_i \partial y_i + Z_i \partial z_i$$

$$= \sum_{i=1}^{n} \sum_{r=1}^{n} \{(X_i \frac{\partial x_i}{\partial q_r} + Y_i \frac{\partial y_i}{\partial q_r} + Z_i \frac{\partial z_i}{\partial q_r}) \partial q_r$$

Interchanging the summation ,we get

$$\sum_{i=1}^{n} (X_i \partial x_i + Y_i \partial y_i + Z_i \partial z_i)$$

$$\sum_{r=1}^{n} \sum_{i=1}^{n} \{X_i \frac{\partial x_i}{\partial q_r} + Y_i \frac{\partial y_i}{\partial q_r} + Z_i \frac{\partial z_i}{\partial q_r}\} \partial q_r$$

$$= \sum_{r=1}^{n} Q_r \partial q_r \text{......... ...(2)}$$

where

$$Q_r = \sum_{i=1}^{n} \left(X_i \frac{\partial x_i}{\partial q_r} + Y_i \frac{\partial y_i}{\partial q_r} + Z_i \frac{\partial z_i}{\partial q_r} \right)$$

and are called the generalised components of external forces.

Now

$$\frac{dx_i}{dt} = \dot{x}_1$$
$$= \frac{\partial x_i}{\partial q_1}\dot{q}_1 + \frac{\partial x_i}{\partial q_2}\dot{q}_2 + \cdots \ldots . + \frac{\partial x_i}{\partial q_r}\dot{q}_r + \cdots . + \frac{\partial x_i}{\partial q_n}\dot{q}_n + \frac{dx_i}{dt}$$

So we have $\frac{\partial \dot{x}_i}{\partial q_r} = \frac{\partial x_i}{\partial q_r}$ **.............(3)**

Also $\frac{\partial}{\partial q_r}\frac{dx_i}{dt} = \frac{d}{dt}\left(\frac{\partial x_i}{\partial q_r}\right)$**................(4)**

Now

$$\ddot{x}_i \partial x_i = \sum_{r=1}^{n} \ddot{x}_i \frac{\partial \dot{x}_i}{\partial q_r} \partial q_r$$

But

$$\ddot{x}_i \frac{\partial x_i}{\partial q_r} = \ddot{x}_i \frac{\partial \dot{x}_i}{\partial \dot{q}_r} = \frac{d}{dt}\left(\dot{x}_i \frac{\partial \dot{x}_i}{\partial \dot{q}_r}\right) - \dot{x}_i \frac{d}{dt}\left(\frac{\partial \dot{x}_i}{\partial \dot{q}_r}\right)$$

from equation (3)

$$= \frac{d}{dt}\left\{\frac{\partial}{\partial \dot{q}_r}\left(\frac{1}{2}\dot{x}_i^{\,2}\right)\right\} - \dot{x}_i \frac{d}{dt}\left(\frac{\partial x_i}{\partial q_r}\right)$$

From equation (3)

$$= \frac{d}{dt}\left\{\frac{\partial}{\partial \dot{q}_r}\left(\frac{1}{2}\dot{x}_i^{\,2}\right)\right\} - \dot{x}_i \frac{d}{dq_r}\left(\frac{\partial x_i}{\partial t}\right)$$

From equation (4)

$$= \frac{d}{dt}\left\{\frac{\partial}{\partial \dot{q}_r}\left(\frac{1}{2}\dot{x}_i^{\,2}\right)\right\} - \dot{x}_i \left(\frac{\partial \dot{x}_i}{\partial q_r}\right)$$

$$\ddot{x}_i \frac{\partial x_i}{\partial q_r} = \frac{d}{dt}\left\{\frac{\partial}{\partial \dot{q}_r}\left(\frac{1}{2}\dot{x}_i^{\,2}\right)\right\} - \frac{\partial}{\partial q_r}\left(\frac{1}{2}\dot{x}_i^2\right)$$

Similarly

$$\ddot{y}_i \frac{\partial y_i}{\partial q_r} = \frac{d}{dt}\left\{\frac{\partial}{\partial \dot{q}_r}\left(\frac{1}{2}\dot{y}_i^{\,2}\right)\right\} - \frac{\partial}{\partial q_r}\left(\frac{1}{2}\dot{y}_i^2\right)$$

$$\ddot{z}_i \frac{\partial z_i}{\partial q_r} = \frac{d}{dt}\left\{\frac{\partial}{\partial \dot{q}_r}\left(\frac{1}{2}\dot{z}_i^{\,2}\right)\right\} - \frac{\partial}{\partial q_r}\left(\frac{1}{2}\dot{z}_i^2\right)$$

Thus L.H.S of equation (1)

$$\sum_{i=1}^{n} m_i \left(\ddot{x}_i \partial x_i + \ddot{y}_i \partial y_i + \ddot{z}_i \partial z_i\right)$$

$$= \sum_{i=1}^{n} m_i \left[\sum_{r=1}^{n} \left\{\ddot{x}_i \frac{\partial x_i}{\partial q_r} + \ddot{y}_i \frac{\partial y_i}{\partial q_r} + \ddot{z}_i \frac{\partial z_i}{\partial q_r}\right\}\right] \partial q_r$$

$$= \sum_{r=1}^{n} \left[\frac{d}{dt}\left\{\frac{\partial}{\partial \dot{q}_r} \sum_{i=1}^{n} \frac{1}{2}\left(\dot{x}_i^2 + \dot{y}_i^2 + \dot{z}_i^2\right)\right\} - \frac{\partial}{\partial q_r}\left\{\sum_{i=1}^{n} \frac{1}{2} m_i\left(\dot{x}_i^2 + \dot{y}_i^2 + \dot{z}_i^2\right)\right\}\right] \partial q_r$$

$$= \sum_{r=1}^{n} \left[\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{q}_r}\right) - \frac{\partial T}{\partial q_r}\right] \partial q_r$$

$$= \sum_{r=1}^{n} Q_r \partial q_r$$

But , the system is holonomic.

So $\partial q_1, \partial q_2 \dots \dots \dots . \partial q_n$ **are arbitrary .**

So taking the corresponding coefficients , we get

$$\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{q}_r}\right) - \frac{\partial T}{\partial q_r} = Q$$

This is called Lagrange's equation of motion for holonomic dynamical system.